CARNEGIE INSTITUTE OF TECHNOLOGY
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THE FORCED OSCILLATION OF SHALLOW DRAFT SHIPS

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Pittsburgh, Pennsylvania

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NOTATION

$\bar{A}$	Area of water plane of ship
$\bar{a}$	Half length of ship
$a = k\bar{a}$	$\sigma^2\bar{a}/g$ or $2\pi\bar{a}/\bar{\lambda}$
$\bar{b}$	Half beam of ship
$\tilde{b} = \bar{b}/\bar{a}$	Ratio of beam to length of ship
$f(\tilde{x},\tilde{z})$	Density of field potential in normalized co-ordinates
G	Green's function for fluid flow problems
g	Acceleration of gravity
$\bar{H}$	Damping factor for translatory motion
$h(x,z)$	Boundary values given on immersed surface of ship
$\bar{I}$	Added moment of inertia
j	Index for components of motion
k	Wave number, σ^2/g
$\bar{M}$	Added mass
$\bar{N}$	Damping factor for rotational motion
$\underline{n}$	Normal to immersed surface of ship
$p(x,y,z)$	Space dependence of dynamic fluid pressure
$\bar{R}$	Distance, $\sqrt{\bar{r}^2 + \bar{y}^2}$
r	Distance, $\sqrt{(\bar{x} - \bar{\xi})^2 + (\bar{z} - \bar{\zeta})^2}$
r_o	Minimum distance between two neighboring pivotal points
$S(\bar{x},\bar{z}:t)$	Immersed surface of ship in motion
$S^{\bullet}(\bar{x},\bar{z})$	Immersed surface of ship at rest
$U(\bar{x},\bar{y},\bar{z})$	Space dependence of field potential
$V(\tilde{x},\tilde{y},\tilde{z})$	Space dependence of field potential in normalized co-ordinates
$w(x,y,z)$	Coefficients of ε^m in series development of $p^m(x,y,z)$
$\bar{X},\bar{Y},\bar{Z}$	Surge, heave, and Sway
$\bar{x},\bar{y},\bar{z}$	Fixed space co-ordinates
$\bar{x}',\bar{y}',\bar{z}'$	Moving space co-ordinates
x,y,z	Normalized co-ordinates by dividing with wave length
$\tilde{x},\tilde{y},\tilde{z}$	Normalized co-ordinates by dividing with half length of ship

NOTATION

α	Perturbation parameter for small motion
ϵ	Perturbation parameter for shallow draft ship
$\bar{\epsilon}$	Maximum draft at center of shallow draft ship
$\bar{\xi},\bar{\eta},\bar{\zeta}$	Co-ordinates of a point on immersed surface of ship
$\Theta_x,\Theta_y,\Theta_z$	Angles of roll, yaw and pitch
ρ	Density of fluid
σ	Frequency of oscillation
ϕ	Velocity potential

Introduction

The forced oscillation of a rigid body in the surface of a fluid is investigated in this paper. John [1] showed that the problem of steady state oscillations of a body in a free surface can be reduced to a Fredholm integral equation. However the solution of the integral equation is difficult except for the case of special cross section geometry. Peters and Stoker [2] as well as Haskind [3] developed a method of solution for a thin ship.

The present paper concerns the three dimensional problem of a ship with small draft. We consider the circular and elliptic disks and determine the dependence of the added mass, added moment of inertia, and damping factor on the frequency of the forced oscillations. A body form of disk provides large wave-making effects so that the results will serve as a complement to the thin ship theory. Two dimensional aspects of this problem have been treated in [4] and [5].

I. General Formulation

We suppose the half-plane $\bar{y} < 0$ to be filled with an incompressible, inviscid fluid, $\bar{y} = 0$ corresponding to a free surface, and $\bar{x}$ and $\bar{z}$ denoting rectangular axes on that surface. It is assumed that all motions of the fluid are irrotational, time-periodic, and small enough so that the problem can be linearized by neglecting squared terms.

We suppose that the fluid motion is produced by a ship which is placed in the surface of the fluid at rest and set into the forceed periodic oscillation. When transients are passed, the resulting fluid motion can be considered to be time-periodic with frequency σ. The irrotationality implies the existence of a velocity potential $\phi(\bar{x},\bar{y},\bar{z}:t)$, and the periodicity in time means,

$$\phi(\bar{x},\bar{y},\bar{z}:t) = \mathrm{Re}[U(\bar{x},\bar{y},\bar{z})\, e^{-i\sigma t}]. \tag{I.1}$$

Next expressing the equation of ship surface in the equilibrium position as, (I.2) $$\bar{y} = S^{\bullet}(\bar{x},\bar{z}),$$

we assume that the form of the ship $S^{\bullet}(\bar{x},\bar{z})$ satisfy the following geometric conditions:

(I.3)
$S^{\bullet}(\bar{x},\bar{z}) = 0$ along the edge of the equilibrium water plane area which is bounded by the curve C,

$S^{\bullet}(0,0) = \bar{\varepsilon}$, maximum draft at the center,

$S^{\bullet}(\bar{x},\bar{z}) = S^{\bullet}(-\bar{x},\bar{z})$, and $S^{\bullet}(\bar{x},\bar{z}) = S^{\bullet}(\bar{x},-\bar{z})$.

In view of the symmetry the co-ordinates of the mass center can be written as $(0,\bar{y}_{\bullet},0)$. If we express the position vector of the mass center as $\underline{R} = \hat{i}\bar{X} + \hat{j}\bar{Y} + \hat{k}\bar{Z}$, its components are,

(I.4)
$$\bar{X}(t) = \mathrm{Re}(\bar{X}^{\bullet}\ e^{-i\sigma t}),$$

$$\bar{Y}(t) = \bar{y} + \mathrm{Re}(\bar{Y}^{\bullet}\ e^{-i\sigma t}),$$

$$\bar{Z}(t) = \mathrm{Re}(\bar{Z}^{\bullet}\ e^{-i\sigma t}).$$

We call $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, the surge, the heave, and the sway, respectively, and $\bar{X}^{\bullet}$, $\bar{Y}^{\bullet}$, and $\bar{Z}^{\bullet}$, the amplitude of the respective motions.

Now let us introduce a set of moving co-ordinates $(\bar{x}',\bar{y}',\bar{z}')$ whose origin is attached to the centroid of water plane of the ship in the equilibrium position. We see that $(\bar{x}',\bar{y}',\bar{z}')$ will coincide with fixed co-ordinates $(\bar{x},\bar{y},\bar{z})$ if the ship is at rest. From rigid body dynamics, the velocity of a particle in the ship at any time is given by,

(I.5) $$\dot{\underline{r}} = \dot{\underline{R}} + \underline{\omega} \times (\underline{r}' - \hat{j}\bar{y}_{\bullet})$$

where $\underline{r} = \hat{i}\bar{x} + \hat{j}\bar{y} + \hat{k}\bar{z}$, $\underline{r}' = \hat{i}\bar{x}' + \hat{j}\bar{y}' + \hat{k}\bar{z}'$, $\underline{\omega} = i\dot{\theta}_x + j\dot{\theta}_y + k\dot{\theta}_z$, and

$$\Theta_x(t) = \mathrm{Re}(\Theta_x^\bullet\, e^{-i\sigma t}),$$

(I.6) $$\Theta_y(t) = \mathrm{Re}(\Theta_y^\bullet\, e^{-i\sigma t}),$$

$$\Theta_z(t) = \mathrm{Re}(\Theta_z^\bullet\, e^{-i\sigma t}).$$

We call Θ_x,Θ_y, and Θ_z, the roll, yaw, and pitch, respectively, and $\Theta_x^\bullet$,$\Theta_y^\bullet$, and $\Theta_z^\bullet$,the amplitude of the respective motions.

In order to show the meaning of small motions, we transform the space variables by,

(I.7) $$x = k\bar{x}, \qquad y = k\bar{y}, \qquad z = k\bar{z},$$

where the wave number k is $k = \sigma^2/g = 2\pi/\bar{\lambda}$, $\bar{\lambda}$ being the wave length of free waves of frequency σ. The normalized amplitudes for the linear motions are given by $X^\bullet = \alpha X'$, $Y^\bullet = \alpha Y'$, and $Z^\bullet = \alpha Z'$, α being a small parameter. We observe that α small means the ratio of actual amplitudes to the wave length is small. Similarly the amplitudes for rotational motions are given by $\Theta_x^\bullet = \alpha\Theta_x'$, $\Theta_y^\bullet = \alpha\Theta_y'$, and $\Theta_z^\circ = \alpha\Theta_z'$.

Neglecting squared terms means that the potential of the fluid motion produced by the forced oscillation will have the form,

(I.8) $$\phi = \alpha\phi',$$

where ϕ' and its derivatives are bounded and we are simply neglecting terms involved α^2 in all our formulations. For instance we express Bernoulli's equation as,

(I.9) $$P = -\rho g\bar{y} - \rho\phi_t(\bar{x},\bar{y},\bar{z}:t) = -\rho g\bar{y} + \rho\,\mathrm{Re}[i\sigma U(\bar{x},\bar{y},\bar{z})\, e^{-i\sigma t}],$$

while the surface elevation is,

(I.10) $$\bar{\eta}(\bar{x},\bar{z}:t) \approx -\frac{1}{g}\,\phi_t(\bar{x},0,\bar{z}:t).$$

We shall now consider the boundary conditions governing the potential function $U(\bar{x},\bar{y},\bar{z})$ in (I.1). As the fluid is assumed to be incompressible and irrotational,

$$\nabla^2\phi = 0,$$

or

$$\nabla^2 U = 0 \qquad \text{in } \bar{y} < 0, \text{ outside the ship.} \tag{I.11}$$

Neglecting terms in α^2, the free surface condition is,

$$\phi_{tt} + g\phi_{\bar{y}} = 0,$$

or

$$U_{\bar{y}} - kU = 0 \text{ on } \bar{y} = 0, \text{ outside the ship.} \tag{I.12}$$

As no flow occurs across the surface of the ship, the kinematic condition can be written as,

$$\phi_n = \underline{n}(\bar{x},\bar{y},\bar{z}:t)\cdot\dot{\underline{r}}(\bar{x},\bar{y},\bar{z}:t) \quad \text{on the immersed surface } \bar{y} = S(\bar{x},\bar{z}:t)$$

where $\underline{n}$ is the normal vector to the surface given by,

$$\underline{n} = \hat{i}\cos(\underline{n},\bar{x}) + \hat{j}\cos(\underline{n},\bar{y}) + \hat{k}\cos(\underline{n},\bar{z}),$$

and $\dot{\underline{r}}$ is the velocity vector of a point $(\bar{x},\bar{y},\bar{z})$ on the surface. We remark here that the immersed surface in motion $S(\bar{x},\bar{z}:t)$ differs from the equlibrium surface $S^{*}(\bar{x},\bar{z})$ introduced earlier. However it is consistent with the omission of terms in α^2 to require this condition to be satisfied on the equilibrium surface.
If we denote the velocity potential for surge, heave, sway, roll, yaw, and pitch by ϕ_j $j = 1,2,\ldots 6$, respectively, the linearity of the problem permits to write the total potential as $\phi = \sum_{j=1}^{6}\phi_j$. Furthermore the components satisfy,

$$(\phi_1)_n = \dot{X}\cos(\underline{n},\bar{x}), \qquad (\phi_2)_n = \dot{Y}\cos(\underline{n},\bar{y}), \qquad (\phi_3)_n = \dot{Z}\cos(\underline{n},\bar{z}),$$

$$(\phi_4)_n = \dot{\theta}_x[(\bar{y}-\bar{y}_o)\cos(\underline{n},\bar{z})-\bar{z}\cos(\underline{n},\bar{y})],$$

$$(\phi_5)_n = \dot{\theta}_y[\bar{z}\cos(\underline{n},\bar{x})-\bar{x}\cos(\underline{n},\bar{z})], \quad (\phi_6)_n = \dot{\theta}_z[\bar{x}\cos(\underline{n},\bar{y})-(\bar{y}-\bar{y}_o)\cos(\underline{n},\bar{x})],$$

hence,

$$(U_1)_n = -i\sigma\dot{X}^{\bullet}\cos(\underline{n},\bar{x}), \quad (U_2)_n = -i\sigma\dot{Y}^{\bullet}\cos(\underline{n},\bar{y}), \quad (U_3)_n = -i\sigma\dot{Z}^{\bullet}\cos(\underline{n},\bar{z}),$$

(I.13)

$$(U_4)_n = -i\sigma\dot{\theta}_x[(\bar{y}-\bar{y}_o)\cos(\underline{n},\bar{z})-\bar{z}\cos(\underline{n},\bar{y})],$$

$$(U_5)_n = -i\sigma\dot{\theta}_y[\bar{z}\cos(\underline{n},\bar{x})-\bar{x}\cos(\underline{n},\bar{z})],$$

$$(U_6)_n = -i\sigma\dot{\theta}_z[\bar{x}\cos(\underline{n},\bar{y})-(\bar{y}-\bar{y}_o)\cos(\underline{n},\bar{x})], \quad \text{on} \quad \bar{y} = S(\bar{x},\bar{z};t).$$

Finally at large distance, the propagating disturbance must have the form of a radially outgoing progressive wave, that is,

(I.14)
$$U(\bar{x},\bar{y},\bar{z}) - \frac{f(\theta)}{\sqrt{\bar{r}}}\, e^{k\bar{y}}\, e^{ik\bar{r}} = O(\tfrac{1}{\bar{r}}) \qquad \text{as } \bar{r} \to \infty,$$

where $\bar{r} = \sqrt{\bar{x}^2 + \bar{z}^2}$, and $\theta = \arctan(\frac{\bar{z}}{\bar{x}})$.

To show clearly the dependence of the solution on parameters, we now introduce the dynamic pressure functions $p_j(x,y,z)$ $j = 1,2,\ldots 6$, by,

(I.15)

$$g\,\frac{X^{\bullet}}{k}\,p_1(x,y,z) = i\sigma\,U_1\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

$$g\,\frac{Y^{\bullet}}{k}\,p_2(x,y,z) = i\sigma\,U_2\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

$$g\,\frac{Z^{\bullet}}{k}\,p_3(x,y,z) = i\sigma\,U_3\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

$$g\,\theta^{\bullet}_x\,\frac{a}{k}\,p_4(x,y,z) = i\sigma\,U_4\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

$$g\,\theta^{\bullet}_y\,\frac{a}{k}\,p_5(x,y,z) = i\sigma\,U_5\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

$$g\,\theta^{\bullet}_z\,\frac{a}{k}\,p_6(x,y,z) = i\sigma\,U_6\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

then the conditions (I.11) - (I.14) can be expressed as,

(I.16) $\nabla^2 p_j = 0$ in $y < 0$, outside the ship.

(I.17) $(p_j)_y - (p_j) = 0$ on $y = 0$, outside the ship,

$$(p_1)_n = \cos(\underline{n},x), \quad (p_2)_n = \cos(\underline{n},y), \quad (p_3)_n = \cos(\underline{n},z),$$

(I.18) $$(p_4)_n = (y-y_0)\cos(\underline{n},z)-z\cos(\underline{n},y),$$

$$(p_5)_n = z\cos(\underline{n},x)-x\cos(\underline{n},z), \quad (p_6)_n = x\cos(\underline{n},y)-(y-y_0)\cos(\underline{n},x)$$

on $y = S(x,z:t)$,

(I.19) $$p_j - \frac{f(0)}{\sqrt{r}} e^{y} e^{ir} = O(\frac{1}{r}) \qquad \text{as } r \to \infty.$$

Equations (I.18) can be rewritten as,

(I.20)
$$(p_1)_n = \frac{-S_x}{\sqrt{1+S_x^2+S_z^2}}, \qquad (p_4)_n = \frac{-(y-y_0)S_z-z}{\sqrt{1+S_x^2+S_z^2}},$$

$$(p_2)_n = \frac{1}{\sqrt{1+S_x^2+S_z^2}}, \qquad (p_5)_n = \frac{-zS_x+xS_z}{\sqrt{1+S_x^2+S_z^2}},$$

$$(p_3)_n = \frac{-S_z}{\sqrt{1+S_x^2+S_z^2}}, \qquad (p_6)_n = \frac{x+(y-y_0)S_x}{\sqrt{1+S_x^2+S_z^2}},$$

on $y = S(x,z:t)$.

Here if we neglect terms in α^2, the quantities $S(x,z:t)$ in (I.20) can be replaced by $S^*(x,z)$.

By (I.15) we have Bernoulli's equation in the form,

$$P_1 = -\rho g \frac{y}{k} + \rho g \frac{X^*}{k} \mathrm{Re}[p_1(x,y,z)\, e^{-i\sigma t}],$$

$$
\text{(I.21)} \qquad
\begin{aligned}
P_2 &= -\rho g \frac{y}{k} + \rho g \frac{Y^\circ}{k} \mathrm{Re}[p_2(x,y,z)\ e^{-i\sigma t}],\\
P_3 &= -\rho g \frac{y}{k} + \rho g \frac{Z^\circ}{k} \mathrm{Re}[p_3(x,y,z)\ e^{-i\sigma t}],\\
P_4 &= -\rho g \frac{y}{k} + \rho g \Theta_x^\circ \frac{a}{k} \mathrm{Re}[p_4(x,y,z)\ e^{-i\sigma t}],\\
P_5 &= -\rho g \frac{y}{k} + \rho g \Theta_y^\circ \frac{a}{k} \mathrm{Re}[p_5(x,y,z)\ e^{-i\sigma t}],\\
P_6 &= -\rho g \frac{y}{k} + \rho g \Theta_z^\circ \frac{a}{k} \mathrm{Re}[p_6(x,y,z)\ e^{-i\sigma t}].
\end{aligned}
$$

II. Shallow Draft Approximation

In this section we relate the first order forces and moments exerted on a ship which oscillates with six degrees of freedom to the added mass, added moment of inertia, and damping factor. Then we develop the perturbation procedure for a ship of small draft.

The three components of the fluid force relative to the fixed co-ordinate system are,

$$
\text{(II.1)} \qquad
\begin{aligned}
F_x &= \iint_S P(\bar{x},\bar{y},\bar{z}:t)\cos(\underline{n},\bar{x})\ dS,\\
F_y &= \iint_S P(\bar{x},\bar{y},\bar{z}:t)\cos(\underline{n},\bar{y})\ dS,\\
F_z &= \iint_S P(\bar{x},\bar{y},\bar{z}:t)\cos(\underline{n},\bar{z})\ dS,
\end{aligned}
$$

where P is the pressure, and S here represents the immersed surface in motion. The three components of moment relative to the fixed co-ordinate axes are,

$$
\text{(II.2)} \qquad
\begin{aligned}
G_x &= \iint_S P[(\bar{y}-\bar{Y})\cos(\underline{n},\bar{z})-(\bar{z}-\bar{Z})\cos(\underline{n},\bar{y})]\ dS,\\
G_y &= \iint_S P[(\bar{z}-\bar{Z})\cos(\underline{n},\bar{x})-(\bar{x}-\bar{X})\cos(\underline{n},\bar{z})]\ dS,\\
G_z &= \iint_S P[(\bar{x}-\bar{X})\cos(\underline{n},\bar{y})-(\bar{y}-\bar{Y})\cos(\underline{n},\bar{x})]\ dS.
\end{aligned}
$$

From the calculated results of forces and moments presented in [1], for the ship satisfying the symmetry conditions given by (I.3) we find,

$$F_x = -\alpha\rho \iint_{S^*} \phi_t^1 \cos(\underline{n},\bar{x})\, dS + O(\alpha^2),$$

$$\text{(II.3)} \qquad F_y = -\alpha\rho \iint_{S^*} \phi_t^1 \cos(\underline{n},\bar{y})\, dS + \bar{M}g - \alpha\rho g \bar{Y}' \bar{A} + O(\alpha^2),$$

$$F_z = -\alpha\rho \iint_{S^*} \phi_t^1 \cos(\underline{n},\bar{z})\, dS + O(\alpha^2),$$

and

$$G_x = \alpha\rho \iint_{S^*} \phi_t^1 [\bar{z}\cos(\underline{n},\bar{y}) - (\bar{y}-\bar{y}_*)\cos(\underline{n},\bar{z})] dS - \alpha\rho g \Theta_x' \iint_A (\bar{z}^2 + \frac{S^{*2}}{2}) d\bar{x} d\bar{z} + O(\alpha^2),$$

$$\text{(II.4)} \qquad G_y = \alpha\rho \iint_{S^*} \phi_t^1 [\bar{x}\cos(\underline{n},\bar{z}) - \bar{z}\cos(\underline{n},\bar{x})] dS + O(\alpha^2),$$

$$G_z = \alpha\rho \iint_{S^*} \phi_t^1 [(\bar{y}-\bar{y}_*)\cos(\underline{n},\bar{x}) - \bar{x}\cos(\underline{n},\bar{y})] dS - \alpha\rho g \Theta_z' \iint_A (\bar{x}^2 + \frac{S^{*2}}{2}) d\bar{x} d\bar{z} + O(\alpha^2),$$

where S^* denotes the immersed surface in the equilibrium position, and A, water plane area of the ship.

Next we express the forces and moments for the oscillatory motion, up to terms of order α, in the dimensionless form,

$$\frac{k^3}{\alpha\rho g X'} \mathrm{Re}(F_x e^{-i\sigma t}) = -\mathrm{Re}[\iint_{S^*} p_1^1 \cos(\underline{n},x) dS\ e^{-i\sigma t}],$$

$$\text{(II.5)} \qquad \frac{k^3}{\alpha\rho g Y'} \mathrm{Re}(F_y e^{-i\sigma t}) = -\mathrm{Re}[\iint_{S^*} p_2^1 \cos(\underline{n},y) dS\ e^{-i\sigma t}] + \frac{k\bar{M}}{\alpha\rho Y'} \mathrm{Re}(e^{-i\sigma t}) - k^2 \bar{A} \mathrm{Re}(e^{-i\sigma t}),$$

$$\frac{k^3}{\alpha\rho g Z'} \mathrm{Re}(F_z e^{-i\sigma t}) = -\mathrm{Re}[\iint_{S^*} p_3^1 \cos(\underline{n},z) dS\ e^{-i\sigma t}],$$

and

$$\frac{k^4}{\alpha\rho g a \Theta_x'} \mathrm{Re}(G_x e^{-i\sigma t}) = \mathrm{Re}[\iint_{S^*} p_4^1 (z\cos(\underline{n},y) - (y-y_*)\cos(\underline{n},z)) dS\ e^{-i\sigma t}] - \frac{1}{k} \iint_A (z^2 + \frac{S^{*2}}{2}) dx dz\ \mathrm{Re}(e^{-i\sigma t}),$$

$$\text{(II.6)} \qquad \frac{k^4}{\alpha\rho g a \Theta_y'} \mathrm{Re}(G_y e^{-i\sigma t}) = \mathrm{Re}[\iint_{S^*} p_5^1 (x\cos(\underline{n},z) - z\cos(\underline{n},x) dS\ e^{-i\sigma t}],$$

$$\frac{k^4}{\alpha\rho g a\Theta_z^{\prime}} \mathrm{Re}(G_z e^{-i\sigma t}) = \mathrm{Re}[\iint_{S^*} p_6^{\prime}((y-y_0)\cos(\underline{n},x) - x\cos(\underline{n},y))dS\ e^{-i\sigma t}]$$

$$-\frac{1}{k}\iint_A (x^2 + \frac{S^{*2}}{2})dxdz\ \mathrm{Re}(e^{-i\sigma t}).$$

Now we seek to express the oscillatory forces and moments as,

$$\mathrm{Re}(F_x e^{-i\sigma t}) = -\bar{M}_x\ddot{X} - \bar{N}_x\dot{X}, \quad \mathrm{Re}(G_x e^{-i\sigma t}) = -\bar{I}_x\bar{a}\ddot{\Theta}_x - \bar{H}_x\bar{a}\dot{\Theta}_x - \rho g\Theta_x\iint_{\bar{A}}(\bar{z}^2 + \frac{S^{*2}}{2})d\bar{x}d\bar{z},$$

$$\text{(II.7)}\quad \mathrm{Re}(F_y e^{-i\sigma t}) = -\bar{M}_y\ddot{Y} - \bar{N}_y\dot{Y} + \bar{M}g\mathrm{Re}(e^{-i\sigma t}) - \rho g\bar{Y}\bar{A}, \quad \mathrm{Re}(G_y e^{-i\sigma t}) = -\bar{I}_y\bar{a}\ddot{\Theta}_y - \bar{H}_y\bar{a}\dot{\Theta}_y,$$

$$\mathrm{Re}(F_z e^{-i\sigma t}) = -\bar{M}_z\ddot{Z} - \bar{N}_z\dot{Z}, \quad \mathrm{Re}(G_z e^{-i\sigma t}) = -\bar{I}_z\bar{a}\ddot{\Theta}_z - \bar{H}_z\bar{a}\dot{\Theta}_z - \rho g\Theta_z\iint_{\bar{A}}(\bar{x}^2 + \frac{S^{*2}}{2})d\bar{x}d\bar{z},$$

where $\bar{M}$ and $\bar{I}$ are called the added mass and added moment of inertia, resepectively, while $\bar{N}$ and $\bar{H}$ are called the damping factors for translation and rotation, respectively.

The substitution of (I.4) and (I.6) in (II.7) yields,

$$\mathrm{Re}(F_x e^{-i\sigma t}) = \alpha\mathrm{Re}[(\sigma^2 M_x + i\sigma N_x)\frac{X^{\prime}}{k} e^{-i\sigma t}],$$

$$\mathrm{Re}(F_y e^{-i\sigma t}) = \alpha\mathrm{Re}[(\sigma^2 M_y + i\sigma N_y)\frac{Y^{\prime}}{k} e^{-i\sigma t}] + \bar{M}g\mathrm{Re}(e^{-i\sigma t}) - \alpha\rho g\mathrm{Re}(\bar{Y}^{\prime} e^{-i\sigma t})A,$$

$$\mathrm{Re}(F_z e^{-i\sigma t}) = \alpha\mathrm{Re}[(\sigma^2 M_z + i\sigma N_z)\frac{Z^{\prime}}{k} e^{-i\sigma t}],$$

(II.8)

$$\mathrm{Re}(G_x e^{-i\sigma t}) = \alpha\mathrm{Re}[(\sigma^2 I_x + i\sigma H_x)\frac{a}{k}\Theta_x^{\prime} e^{-i\sigma t}] - \alpha\rho g\mathrm{Re}(\Theta_x^{\prime}\frac{a}{k}e^{-i\sigma t})\iint_A(\bar{z}^2 + \frac{S^{*2}}{2})d\bar{x}d\bar{z},$$

$$\mathrm{Re}(G_y e^{-i\sigma t}) = \alpha\mathrm{Re}[(\sigma^2 I_y + i\sigma H_y)\frac{a}{k}\Theta_y^{\prime} e^{-i\sigma t}],$$

$$\mathrm{Re}(G_z e^{-i\sigma t}) = \alpha\mathrm{Re}[(\sigma^2 I_z + i\sigma H_z)\frac{a}{k}\Theta_z^{\prime} e^{-i\sigma t}] - \alpha\rho g\mathrm{Re}(\Theta_z^{\prime}\frac{a}{k}e^{-i\sigma t})\iint_A(\bar{x}^2 + \frac{S^{*2}}{2})d\bar{x}d\bar{z},$$

hence in the dimensionless form,

$$\frac{k^3}{\alpha\rho\sigma X^{\prime}}\mathrm{Re}(F_x e^{-i\sigma t}) = k^3\mathrm{Re}[(\frac{\bar{M}_x}{\rho} + i\frac{\bar{N}_x}{\rho\sigma})\ e^{-i\sigma t}],$$

$$\frac{k^3}{\alpha\rho g Y'} \mathrm{Re}(F_y e^{-i\sigma t}) = k^3 \mathrm{Re}[(\frac{\bar{M}_y}{\rho} + i\frac{\bar{N}_y}{\rho\sigma}) e^{-i\sigma t}] + \frac{k^3\bar{M}}{\alpha\rho g Y'} \mathrm{Re}(e^{-i\sigma t}) - k^2\bar{A}\mathrm{Re}(e^{-i\sigma t}),$$

$$\frac{k^3}{\alpha\rho g Z'} \mathrm{Re}(F_z e^{-i\sigma t}) = k^3 \mathrm{Re}[(\frac{\bar{M}_z}{\rho} + i\frac{\bar{N}_z}{\rho\sigma}) e^{-i\sigma t}],$$

$$\text{(II.9)} \quad \frac{k^4}{\alpha\rho g a\Theta'_x} \mathrm{Re}(G_x e^{-i\sigma t}) = k^4 \mathrm{Re}[(\frac{\bar{I}_x}{\rho} + i\frac{\bar{H}_x}{\rho\sigma}) e^{-i\sigma t}] - \frac{1}{k}\iint_A (z^2 + \frac{S^{*2}}{2}) dx dz \mathrm{Re}(e^{-i\sigma t}),$$

$$\frac{k^4}{\alpha\rho g a\Theta'_y} \mathrm{Re}(G_y e^{-i\sigma t}) = k^4 \mathrm{Re}[(\frac{\bar{I}_y}{\rho} + i\frac{\bar{H}_y}{\rho\sigma}) e^{-i\sigma t}],$$

$$\frac{k^4}{\alpha\rho g a\Theta'_z} \mathrm{Re}(G_z e^{-i\sigma t}) = k^4 \mathrm{Re}[(\frac{\bar{I}_z}{\rho} + i\frac{\bar{H}_z}{\rho\sigma}) e^{-i\sigma t}] - \frac{1}{k}\iint_A (x^2 + \frac{S^{*2}}{2}) dx dz \mathrm{Re}(e^{-i\sigma t}).$$

Therefore, equating (II.5) and (II.6) to (II.9) we obtain,

$$k^3\bar{M}_x/\rho = -\iint_{S^*} (p_1^1)_r \cos(\underline{n},x) dS, \qquad k^3\bar{N}_x/\rho\sigma = -\iint_{S^*} (p_1^1)_i \cos(\underline{n},x) dS,$$

$$k^3\bar{M}_y/\rho = -\iint_{S^*} (p_2^1)_r \cos(\underline{n},y) dS, \qquad k^3\bar{N}_y/\rho\sigma = -\iint_{S^*} (p_2^1)_i \cos(\underline{n},y) dS,$$

$$k^3\bar{M}_z/\rho = -\iint_{S^*} (p_3^1)_r \cos(\underline{n},z) dS, \qquad k^3\bar{N}_z/\rho\sigma = -\iint_{S^*} (p_3^1)_i \cos(\underline{n},z) dS,$$

$$\text{(II.10)} \quad k^4\bar{I}_x/\rho = \iint_{S^*} (p_4^1)_r [z\cos(\underline{n},y) - (y-y_*)\cos(\underline{n},z)] dS,$$

$$k^4\bar{H}_x/\rho\sigma = \iint_{S^*} (p_4^1)_i [z\cos(\underline{n},y) - (y-y_*)\cos(\underline{n},z)] dS,$$

$$k^4\bar{I}_y/\rho = \iint_{S^*} (p_5^1)_r [x\cos(\underline{n},z) - z\cos(\underline{n},x)] dS,$$

$$k^4\bar{H}_y/\rho\sigma = \iint_{S^*} (p_5^1)_i [x\cos(\underline{n},z) - z\cos(\underline{n},x)] dS,$$

$$k^4\bar{I}_z/\rho = \iint_{S^*} (p_6^1)_r [(y-y_*)\cos(\underline{n},x) - x\cos(\underline{n},y)] dS,$$

$$k^4\bar{H}_z/\rho\sigma = \iint_{S^*} (p_6^1)_i [(y-y_*)\cos(\underline{n},x) - x\cos(\underline{n},y)] dS,$$

where the pressure function $p(x,y,z)$ is resolved to the real and imaginary parts as $p_j = (p_j)_r + i(p_j)_i \qquad j = 1,2,\dots 6.$

We proceed to consider the shallow draft approximation. First we remark that the smallness of draft means the ratio of actual draft to the wave length is small. Therefore, all quantities are to be developed as a power series in $\epsilon = 2\pi\bar{\epsilon}/\bar{\lambda}$.

Let the profile of a ship of small draft be,

$$\text{(II.11)} \qquad y = \epsilon S^1(x,z) \qquad -a \le x \le a, \quad -b \le z \le b.$$

Here $S^1(x,z)$ satisfies the following geometric conditions:

$$S^1(0,0) = 1, \qquad S^1(x,z) = S^1(-x,z), \qquad S^1(x,z) = S^1(x,-z).$$

Since $\sqrt{1+S_x^2+S_z^2} = 1 - \frac{\epsilon^2}{2}[(S_x^1)^2+(S_z^1)^2] + \sum_{m=2}^{\infty} (-1)^m \epsilon^{2m} \alpha_m [(S_x^1)^2+(S_z^1)^2]^m,$

we obtain,

$$\frac{\partial}{\partial n} = [-\epsilon S_x^1 + O(\epsilon^2)] \frac{\partial}{\partial x} + [\,1 + O(\epsilon^2)] \frac{\partial}{\partial y} + [-\epsilon S_z^1 + O(\epsilon^2)] \frac{\partial}{\partial z}.$$

If we call the right hand members of (I.20) $R_j(x,z) \quad j = 1,2,\dots 6,$ these can be expanded in the form,

$$R_j(x,z) = \sum_{m=0}^{\infty} \epsilon^m R_j^m(x,z).$$

The first few terms are,

$$\text{(II.12)} \qquad \begin{aligned} &R_1^0(x,z) = 0, \quad R_2^0(x,z) = 1, \quad R_3^0(x,z) = 0, \quad R_4^0(x,z) = -z, \\ &R_5^0(x,z) = 0, \quad R_6^0(x,z) = x, \quad \dots \;. \end{aligned}$$

Next let us assume that the pressure can be expanded in the form,

$$\text{(II.13)} \qquad p_j(x,y,z) = \sum_{m=0}^{\infty} \epsilon^m p_j^m(x,y,z) \qquad j = 1,2,\dots 6.$$

By Taylor's theorem the function p_j^m can be expressed as,

$$p_j^m[x,\varepsilon S'(x,z),z] = \sum_{n=0}^{\infty} \frac{1}{n!} [\varepsilon S'(x,z)]^n \frac{\partial^n}{\partial y^n} p_j^m(x,0,z).$$

Now we can write the left hand members of (I.20) up to order ε as,

$$[-\varepsilon S'_x + O(\varepsilon^2)] \sum_{m=0}^{1} \sum_{n=0}^{1} \frac{1}{n!} \varepsilon^{m+n} [S'(x,z)]^n \frac{\partial^{n+1}}{\partial x \partial y^n} p_j^m(x,0,z)$$

(II.14) $$+ [\, 1 + O(\varepsilon^2)] \sum_{m=0}^{1} \sum_{n=0}^{1} \frac{1}{n!} \varepsilon^{m+n} [S'(x,z)]^n \frac{\partial^{n+1}}{\partial y^{n+1}} p_j^m(x,0,z)$$

$$+ [-\varepsilon S'_z + O(\varepsilon^2)] \sum_{m=0}^{1} \sum_{n=0}^{1} \frac{1}{n!} \varepsilon^{m+n} [S'(x,z)]^n \frac{\partial^{n+1}}{\partial z \partial y^n} p_j^m(x,0,z)$$

$$= R_j^\circ(x,z) + \varepsilon R_j^1(x,z) \qquad \text{on} \quad y = \varepsilon S'(x,z).$$

Equating the coefficient of the like powers of ε we obtain,

$$\frac{\partial p_j^\circ(x,0,z)}{\partial y} = R_j^\circ(x,z)$$

$$\frac{\partial p_j^1(x,0,z)}{\partial y} = R_j^1(x,z) + S'_x \frac{\partial p_j^\circ(x,0,z)}{\partial x} + S'_z \frac{\partial p_j^\circ(x,0,z)}{\partial z} - S' \frac{\partial^2 p_j^\circ(x,0,z)}{\partial y^2} .$$

More generally the boundary condition for p_j^n is of the form,

$$\frac{\partial p_j^n(x,0,z)}{\partial y} = R_j^n(x,z) + E_j^n \qquad \text{within the water plane of the ship.}$$

Here E_j^n consists of p_j^k for $k \leq n-1$ and their derivatives evaluated at $y = 0$.

Therefore p_j^n can be determined recursively by solving boundary value problems of the following form:

Find a function $w(x,y,z)$ such that,

(II.15) $$\nabla^2 w = 0 \qquad \text{in} \quad y < 0, \text{ outside the ship,}$$

(II.16) $\qquad w_y - w = 0 \qquad$ on $y = 0$, outside the ship,

(II.17) $\qquad w_y = h(x,z) \qquad$ on $y = 0$, within the hull of the ship,

(II.18) $\qquad w - \frac{f(\Theta)}{\sqrt{r}} e^{ky} e^{ikr} = O(\frac{1}{r}) \qquad$ as $r \to \infty$.

Note that $h(x,z)$ is a given function, and for $w = p_j^*$ in particular it is given by (II.12).

We again make a change of independent and dependent variables by introducing,

$$\tilde{x} = x / a, \qquad \tilde{y} = y / a, \qquad \tilde{z} = z / a,$$

(II.19) $\qquad w(x,y,z) = a\, V(x/a,\ y/a,\ z/a) \qquad$ where $\quad a = k\bar{a}$.

Then the boundary value problem becomes:
Find a function $V(\tilde{x},\tilde{y},\tilde{z})$ such that,

(II.20) $\qquad \nabla^2 V = 0 \qquad$ in $\tilde{y} < 0$, outside the ship,

(II.21) $\qquad V_{\tilde{y}} - a\,V = 0 \quad$ on $\tilde{y} = 0$, outside the ship,

(II.22) $\qquad V_{\tilde{y}} = h(\tilde{x},\tilde{z}) \quad$ on $\tilde{y} = 0$, within the hull of the ship,

(II.23) $\qquad V - \frac{f(\Theta)}{\sqrt{\tilde{r}}} e^{a\tilde{y}} e^{ia\tilde{r}} = O(\frac{1}{\tilde{r}}) \quad$ as $\tilde{r} \to \infty$.

Here a comment on the change of variables is in order. Since $x = k\bar{x}$ and $a = k\bar{a}$, the $\tilde{x}$ corresponds to the ratio of actual co-ordinate $\bar{x}$ to the half length of the ship $\bar{a}$. Thus in the new boundary value problem all length dimensions have been made dimensionless by dividing with the half length of the ship instead of the wave length.

Now we turn to the determination of the added mass or added moment of inertia and the damping factor for a ship of small draft. Since we now have $\quad \bar{y}_0 = O(\varepsilon), \quad \cos(\underline{n},x) = -\varepsilon S_x^1 + O(\varepsilon^2), \quad \cos(\underline{n},y) = 1 + O(\varepsilon^2)$, and $\cos(\underline{n},z) = -\varepsilon S_z^1 + O(\varepsilon^2)$, the following expressions can be found from (II.10).

Note that for convenience we adopt the notation $p_j(x,y,z)$ for previously introduced $p_j^1(x,y,z)$. S° is in this case a region in the x-z plane.

$$k^3\bar{M}_x/\rho = -\iint_{S^\circ}[p_1(x,y,z)]_r\cos(\underline{n},x)dS$$

$$= \iint_{S^\circ}\left\{[p_1^\circ(x,z)]_r + \varepsilon S'\frac{\partial[p_1^\circ(x,z)]_r}{\partial y} + \varepsilon[p_1^1(x,z)]_r\right\}\varepsilon S_x^1 dS = O(\varepsilon),$$

similarly $$k^3\bar{N}_x/\rho\sigma = -\iint_{S^\circ}[p_1(x,y,z)]_i\cos(\underline{n},x)dS = O(\varepsilon),$$

$$k^3\bar{M}_y/\rho = -\iint_{S^\circ}[p_2(x,y,z)]_r\cos(\underline{n},y)dS$$

$$= -\iint_{S^\circ}\left\{[p_2^\circ(x,z)]_r + \varepsilon S'\frac{\partial[p_2^\circ(x,z)]_r}{\partial y} + \varepsilon[p_2^1(x,z)]_r\right\}dS = -\iint_{S^\circ}[p_2^\circ(x,z)]_r dS,$$

similarly $$k^3\bar{N}_y/\rho\sigma = -\iint_{S^\circ}[p_2(x,y,z)]_i\cos(\underline{n},y)dS = -\iint_{S^\circ}[p_2^\circ(x,z)]_i dS,$$

$$k^3\bar{M}_z/\rho = -\iint_{S^\circ}[p_3(x,y,z)]_r\cos(\underline{n},z)dS$$

$$= \iint_{S^\circ}\left\{[p_3^\circ(x,z)]_r + \varepsilon S'\frac{\partial[p_3^\circ(x,z)]_r}{\partial y} + \varepsilon[p_3^1(x,z)]_r\right\}\varepsilon S_z^1 dS = O(\varepsilon),$$

similarly $$k^3\bar{N}_z/\rho\sigma = -\iint_{S^\circ}[p_3(x,y,z)]_i\cos(\underline{n},z)dS = O(\varepsilon),$$

(II.24)

$$k^4\bar{I}_x/\rho = \iint_{S^\circ}[p_4(x,y,z)]_r[z\cos(\underline{n},y) - y\cos(\underline{n},z)]dS$$

$$= \iint_{S^\circ}[p_4^\circ(x,z)]_r(\ z + \varepsilon S'\varepsilon S_z^1)dS = \iint_{S^\circ} z[p_4^\circ(x,z)]_r dS,$$

$$k^4\bar{H}_x/\rho\sigma = \iint_{S^\circ}[p_4(x,y,z)]_i[z\cos(\underline{n},y) - y\cos(\underline{n},z)]dS = \iint_{S^\circ} z[p_4^\circ(x,z)]_i dS,$$

$$k^4\bar{I}_y/\rho = \iint_{S^\circ}[p_5(x,y,z)]_r[x\cos(\underline{n},z) - z\cos(\underline{n},x)]dS$$

$$= \iint_{S^\circ}[p_5^\circ(x,z)]_r(-x\varepsilon S' + z\varepsilon S')dS = O(\varepsilon),$$

$$k^4\bar{H}_y/\rho\sigma = \iint_{S^\circ}[p_5(x,y,z)]_i[x\cos(\underline{n},z) - z\cos(\underline{n},x)]dS = O(\varepsilon),$$

$$k^4\bar{I}_z/\rho = \iint_{S^\circ}[p_6(x,y,z)]_r[y\cos(\underline{n},x)-x\cos(\underline{n},y)]dS$$

$$= \iint_{S^\circ}[p_6^\circ(x,z)]_r(-\varepsilon S'\varepsilon S'_x - x)dS = -\iint_{S^\circ} x[p_6^\circ(x,z)]_r dS,$$

$$k^4\bar{H}_z/\rho\sigma = \iint_{S^\circ}[p_6(x,y,z)]_i[y\cos(\underline{n},x)-x\cos(\underline{n},y)]dS = -\iint_{S^\circ} x[p_6^\circ(x,z)]_i dS.$$

Hence, in the dimensionless form we obtain the added mass and damping factor for heave as,

$$\text{(II.25)} \qquad \begin{aligned} M_y &= \bar{M}_y/\rho\bar{a}^3 = -\frac{1}{a}\iint_{\tilde{S}}[p_2^\circ(\tilde{x},\tilde{z})]_r dS, \\ N_y &= \bar{N}_y/\rho\bar{a}^3\sigma = -\frac{1}{a}\iint_{\tilde{S}}[p_2^\circ(\tilde{x},\tilde{z})]_i dS, \end{aligned}$$

the added moment of inertia and damping factor for roll as,

$$\text{(II.26)} \qquad \begin{aligned} I_x &= \bar{I}_x/\rho\bar{a}^4 = \frac{1}{a}\iint_{\tilde{S}} \tilde{z}\,[p_4^\circ(\tilde{x},\tilde{z})]_r dS, \\ H_x &= \bar{H}_x/\rho\bar{a}^4\sigma = \frac{1}{a}\iint_{\tilde{S}} \tilde{z}\,[p_4^\circ(\tilde{x},\tilde{z})]_i dS, \end{aligned}$$

and the added moment of inertia and damping factor for pitch as,

$$\text{(II.27)} \qquad \begin{aligned} I_z &= \bar{I}_z/\rho\bar{a}^4 = -\frac{1}{a}\iint_{\tilde{S}} \tilde{x}\,[p_6^\circ(\tilde{x},\tilde{z})]_r dS, \\ H_z &= \bar{H}_z/\rho\bar{a}^4\sigma = -\frac{1}{a}\iint_{\tilde{S}} \tilde{x}\,[p_6^\circ(\tilde{x},\tilde{z})]_i dS, \end{aligned}$$

where $\tilde{S}$ is the image of S° under the transformation $\tilde{x} = x/a$, $\tilde{z} = z/a$.

By (II.19) we write, $\quad p_j^\circ(x,y,z) = aV^j(x/a,y/a,z/a) \quad j = 2,4,6,$

then from (II.22) we find,

$$\text{(II.28)} \qquad V^2_{\tilde{y}}(\tilde{x},0,\tilde{z}) = 1, \quad V^4_{\tilde{y}}(\tilde{x},0,\tilde{z}) = \tilde{z}, \quad V^6_{\tilde{y}}(\tilde{x},0,\tilde{z}) = \tilde{x} \quad \text{on } \tilde{S},$$

and from (II.21) we have,

$$\text{(II.29)} \qquad V^j_{\tilde{y}}(\tilde{x},0,\tilde{z}) - aV^j(\tilde{x},0,\tilde{z}) = 0 \qquad j = 2,4,6, \text{ outside } \tilde{S}.$$

III. Integral Representation.

We will discuss here the integral representation of the solution of boundary value problems for a surface obstacle of negligible draft. Consider a region bounded by the surface of the ship and the free surface. We shall show that for some function $f(\tilde{x},\tilde{z})$ defined over the surface of the ship, $\tilde{S}$, the potential at any point $(\tilde{x},\tilde{y},\tilde{z})$ in the region is given by,

$$\text{(III.1)} \qquad V(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{4\pi}\iint_{\tilde{S}} f(\tilde{\xi},\tilde{\zeta})\, G(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},0,\tilde{\zeta})\, d\tilde{\xi}d\tilde{\zeta}\,.$$

Here $G(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},0,\tilde{\zeta})$ represent the Green's function given in [1] evaluated at $\tilde{\eta} = 0$, which can be expressed as,

$$\text{(III.2)} \qquad G(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},0,\tilde{\zeta}) = \frac{1}{\tilde{R}} + \int_0^\infty \frac{\beta+a}{\beta-a}\, e^{\beta\tilde{y}}\, J_0(\beta\tilde{r})\, d\beta,$$

$$= \frac{2}{\tilde{R}} + \int_0^\infty \frac{2a}{\beta-a}\, e^{\beta\tilde{y}}\, J_0(\beta\tilde{r})\, d\beta,$$

where the integral sign $\int_0^\infty$ is to be understood as integration along the positive real axis except for an arc in the lower half plane to avoid the positive real root $\beta = a$ of the denominator. Also, $\tilde{r} = \sqrt{(\tilde{x}-\tilde{\xi})^2+(\tilde{z}-\tilde{\zeta})^2}$, and $\tilde{R} = \sqrt{\tilde{r}^2+\tilde{y}^2}$, hence $\tilde{R}$ denotes the distance from a point $(\tilde{x},\tilde{y},\tilde{z})$ in the region to a point $(\tilde{\xi},0,\tilde{\zeta})$ on the surface of the ship. $J_0(\beta\tilde{r})$ is the zero order Bessel function of the first kind.

From (III.2) we see that,

$$\text{(III.3)} \qquad G_{\tilde{y}} - a\,G = \frac{\partial}{\partial\tilde{y}}\frac{2}{\tilde{R}} \qquad \text{in } \tilde{y} < 0.$$

Now we define the following integrals,

$$W(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{4\pi}\iint_{\tilde{S}} f(\tilde{\xi},\tilde{\zeta})\, \frac{2}{\tilde{R}}\, d\tilde{\xi}\, d\tilde{\zeta}\,,$$

$$L(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{4\pi}\iint_{\tilde{S}} f(\tilde{\xi},\tilde{\zeta})\, H(\tilde{x},\tilde{y},\tilde{z})\, d\tilde{\xi}\, d\tilde{\zeta},$$

where

$$H(\tilde{x},\tilde{y},\tilde{z}) = \oint_0^\infty \frac{2a}{\beta - a} J_o(\beta\tilde{r})\, e^{\beta\tilde{y}}\, d\beta$$

It follows from (III.1) that $V(\tilde{x},\tilde{y},\tilde{z}) = W(\tilde{x},\tilde{y},\tilde{z}) + L(\tilde{x},\tilde{y},\tilde{z})$.
For a function $f(\tilde{x},\tilde{z})$ continuous on $\tilde{S}$ the integral L satisfies the condition (II.20), while W + L satisfies the condition (II.23).
From (III.3) we have,

$$V_{\tilde{y}}(\tilde{x},\tilde{y},\tilde{z}) - aV(\tilde{x},\tilde{y},\tilde{z}) = W_{\tilde{y}}(\tilde{x},\tilde{y},\tilde{z}) \qquad \text{in} \quad \tilde{y} < 0.$$

By a theorem of the potential theory,

$$\lim_{\substack{\tilde{y} \to 0 \\ \tilde{y} < 0}} W_{\tilde{y}}(\tilde{x},\tilde{y},\tilde{z}) = \begin{cases} f(\tilde{x},\tilde{z}) & \text{on } \tilde{S}, \\ 0 & \text{outside } \tilde{S}, \end{cases}$$

hence,

(III.4) $$V_{\tilde{y}}(\tilde{x},0,\tilde{z}) - aV(\tilde{x},0,\tilde{z}) = f(\tilde{x},\tilde{z}) \qquad \text{on } \tilde{S},$$

(III.5) $$V_{\tilde{y}}(\tilde{x},0,\tilde{z}) - aV(\tilde{x},0,\tilde{z}) = 0 \qquad \text{outside } \tilde{S}.$$

From (II.22) it can be seen that the potential $V(\tilde{x},\tilde{y},\tilde{z})$ given by (III.1) will be the solution of our boundary value problem if $f(\tilde{x},\tilde{z})$ is chosen as a solution of the following integral equation,

$$h(\tilde{x},\tilde{z}) - aV(\tilde{x},0,\tilde{z}) = f(\tilde{x},\tilde{z}),$$

or,

(III.6) $$f(\tilde{x},\tilde{z}) + \frac{a}{4\pi}\iint_{\tilde{S}} f(\tilde{\xi},\tilde{\zeta})\, G(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta})\, d\tilde{\xi}\, d\tilde{\zeta} = h(\tilde{x},\tilde{z}).$$

Observe that the zero draft approximation leads to the kernel being G itself rather than the normal derivative of G as in [1].

Next we calculate the kernel of the integral equation explicitly.

From (III.2) we have,

$$\text{(III.7)} \qquad G(\tilde{x},0,\tilde{z},\tilde{\xi},0,\tilde{\zeta}) = \frac{2}{\tilde{r}} + 2a \fint_0^\infty \frac{J_0(\beta\tilde{r})}{\beta - a}\, d\beta$$

$$= \frac{2}{\tilde{r}} + 2a \fint_0^\infty \frac{\beta}{\beta^2 - a^2} J_0(\beta\tilde{r})\, d\beta + 2a^2 \fint_0^\infty \frac{J_0(\beta\tilde{r})}{\beta^2 - a^2}\, d\beta.$$

Let us express (III.7) as,

$$\text{(III.8)} \qquad G(\tilde{x},0,\tilde{z},\tilde{\xi},0,\tilde{\zeta}) = \frac{2}{\tilde{r}} + 2aI_1 + 2a^2 I_2,$$

where

$$I_1 = \fint_0^\infty \frac{\beta J_0(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta, \quad \text{and} \quad I_2 = \fint_0^\infty \frac{J_0(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta.$$

We have

$$\text{(III.9)} \qquad \fint_0^\infty \frac{\beta J_0(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta = \int_0^{a-\varepsilon} \frac{\beta J_0(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta + i\frac{\pi}{2} J_0(a\tilde{r}) + \int_{a+\varepsilon}^\infty \frac{\beta J_0(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta$$

$$= i\frac{\pi}{2} J_0(a\tilde{r}) + \mathrm{Re}\left[\int_0^{a-\varepsilon} \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta + \int_{a+\varepsilon}^\infty \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta\right],$$

for small positive ε.

Next let us consider the integral $\fint_0^\infty \beta H_0^{(1)}(\beta\tilde{r})/(\beta-a)(\beta+a)d\beta$ integrated along the real axis except for an arc running above the root $\beta = a$, then we obtain,

$$\text{(III.10)} \qquad \fint_0^\infty \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta = \int_0^{a-\varepsilon} \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta + i\frac{\pi}{2} H_0^{(1)}(a\tilde{r}) + \int_{a+\varepsilon}^{o} \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta$$

$$= i\frac{\pi}{2} H_0^{(1)}(a\tilde{r}) + \int_0^{a-\varepsilon} \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta + \int_{a+\varepsilon}^{o} \frac{\beta H_0^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)}\, d\beta$$

From (III.9) and (III.10) we find,

$$\text{(III.11)} \qquad I_1 = i\frac{\pi}{2} J_o(a\tilde{r}) + \mathrm{Re}[i\frac{\pi}{2} H_o^{(1)}(a\tilde{r}) + \fint_o^\infty \frac{\beta H_o^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)} d\beta]$$

$$= i\frac{\pi}{2} J_o(a\tilde{r}) - \frac{\pi}{2} Y_o(a\tilde{r}) + \mathrm{Re} \fint_o^\infty \frac{\beta H_o^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)} d\beta.$$

Here we can deform the path of integration into the positive imaginary axis by setting $\beta = i\gamma$, then I_1 becomes,

$$I_1 = i\frac{\pi}{2} J_o(a\tilde{r}) - \frac{\pi}{2} Y_o(a\tilde{r}) + \mathrm{Re} \int_o^\infty \frac{\gamma H_o^{(1)}(i\gamma\tilde{r})}{\gamma^2 + a^2} d\gamma$$

$$= i\frac{\pi}{2} J_o(a\tilde{r}) - \frac{\pi}{2} Y_o(a\tilde{r}) - \mathrm{Re}[\frac{2}{\pi} \int_o^\infty \frac{i\gamma K_o(\gamma\tilde{r})}{\gamma^2 + a^2} d\gamma].$$

Observing that the integrand is purely imaginary we now express I_1 as,

$$\text{(III.12)} \qquad I_1 = \frac{\pi}{2}[iJ_o(a\tilde{r}) - Y_o(a\tilde{r})].$$

If the path of integration runs below the root $\beta = a$, for I_2 we obtain,

$$\text{(III.13)} \qquad I_2 = i\frac{\pi}{2a} J_o(a\tilde{r}) + \mathrm{Re}[\int_o^{a-\varepsilon} \frac{H_o^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)} d\beta + \int_{a+\varepsilon}^\infty \frac{H_o^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)} d\beta].$$

Repeating the same process employed to obtain (III.10) we find,

$$\text{(III.14)} \qquad I_2 = i\frac{\pi}{2a} J_o(a\tilde{r}) - \frac{\pi}{2a} Y_o(a\tilde{r}) + \mathrm{Re} \fint_o^\infty \frac{H_o^{(1)}(\beta\tilde{r})}{(\beta-a)(\beta+a)} d\beta.$$

When the path is deformed into the positive imaginary axis, I_2 becomes,

$$I_2 = i\frac{\pi}{2a} J_o(a\tilde{r}) - \frac{\pi}{2a} Y_o(a\tilde{r}) - \mathrm{Re}[\frac{2}{\pi} \int_o^\infty \frac{K_o(\gamma\tilde{r})}{\gamma^2 + a^2} d\gamma].$$

Therefore by a result in [7] we find,

$$\text{(III.15)} \qquad I_2 = i\frac{\pi}{2a} J_o(a\tilde{r}) - \frac{\pi}{2a} Y_o(a\tilde{r}) - \frac{\pi}{2a} [S_o(a\tilde{r}) - Y_o(a\tilde{r})]$$

$$= \frac{\pi}{2a} [iJ_o(a\tilde{r}) - S_o(a\tilde{r})].$$

Finally from (III.12) and (III.15) we obtain,

$$\text{(III.16)} \qquad G(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta}) = \frac{2}{\tilde{r}} + 2a\,\frac{\pi}{2}\,[iJ_o(a\tilde{r})-Y_o(a\tilde{r})] + 2a^2\frac{\pi}{2a}\,[iJ_o(a\tilde{r})-S_o(a\tilde{r})]$$

$$= \frac{2}{\tilde{r}} - \pi a[Y_o(a\tilde{r}) + S_o(a\tilde{r}) - i2J_o(a\tilde{r})],$$

where $Y_o(a\tilde{r})$ denotes the zero order Bessel function of the second kind, and $S_o(a\tilde{r})$, the zero order Struve function, respectively. Thus, the kernel of the integral equation can be evaluated explicitly and indeed this is another reason for introducing the shallow draft approximation. Here we observe that as $a\tilde{r} \to \infty$ $S_o(a\tilde{r}) \approx Y_o(a\tilde{r})$, hence the Green's function given by (III.16) becomes,

$$G(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta}) \approx \frac{2}{\tilde{r}} - 2\pi a[Y_o(a\tilde{r}) - J_o(a\tilde{r})]$$

$$\approx \frac{2}{\tilde{r}} - 2\pi a\left[\frac{\sin(a\tilde{r}-\pi/4)}{\sqrt{\pi(a\tilde{r})/2}} - i\,\frac{\cos(a\tilde{r}-\pi/4)}{\sqrt{\pi(a\tilde{r})/2}}\right]$$

$$\frac{2}{\tilde{r}} + i2\sqrt{\frac{2\pi a}{\tilde{r}}}\; e^{i(a\tilde{r}-\pi/4)} .$$

According to the Fredholm theory the integral equation (III.6) will be soluble if the corresponding homogeneous equation, that is,

$$\text{(III.17)} \qquad f^*(\tilde{x},\tilde{z}) + \frac{a}{4\pi}\iint_{\tilde{S}} f^*(\tilde{\xi},\tilde{\zeta})G(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta})\,d\tilde{\xi}\,d\tilde{\zeta} = 0$$

has only the trivial solution, $f^*(\tilde{x},\tilde{z}) = 0$.
It has been shown in [1] that if f^* is a solution of (III.17) then

$$V^*(\tilde{x},\tilde{y},\tilde{z}) = \frac{1}{4\pi}\iint_{\tilde{S}} f^*(\tilde{\xi},\tilde{\zeta})\; G(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},0,\tilde{\zeta})\,d\tilde{\xi}\,d\tilde{\zeta},$$

vanishes identially in $y \le 0$.

Since $V^{\circ}(\tilde{x},\tilde{y},\tilde{z})$ vanishes, by (III.4) $f^{\circ}(\tilde{x},\tilde{z})$ must also vanish. Hence the integral equation (III.6) is soluble, and we shall have a solution for our boundary value problem in the form of (III.1).

When the solution of the integral equation $f(\tilde{x},\tilde{z})$ is found, we want to determine the added mass and added moment of inertia as well as the damping factors. By substituting (II.28) in (III.4) we find the pressures are related to the density function $f(\tilde{x},\tilde{z})$ as,

$$p_2^{\circ} = aV^2(\tilde{x},0,\tilde{z}) = 1 - f^2(\tilde{x},\tilde{z}) \qquad \text{for heave,} \tag{III.31}$$

$$p_4^{\circ} = aV^4(\tilde{x},0,\tilde{z}) = \tilde{z} - f^4(\tilde{x},\tilde{z}) \qquad \text{for roll,} \tag{III.32}$$

$$p_6^{\circ} = aV^6(\tilde{x},0,\tilde{z}) = \tilde{x} - f^6(\tilde{x},\tilde{z}) \qquad \text{for pitch,} \tag{III.33}$$

where $\quad f^j(\tilde{x},\tilde{z}) = f_r^j(\tilde{x},\tilde{z}) + i\, f_i^j(\tilde{x},\tilde{z}) \qquad j = 2,4,6.$

Therefore we obtain from (III.31) and (II.25),

$$\text{(III.34)} \qquad \begin{aligned} M_y &= -\frac{1}{a}\iint_{\tilde{S}} [\, 1 - f_r^2(\tilde{x},\tilde{z})]\, dS, \\ N_y &= \frac{1}{a}\iint_{\tilde{S}} f_i^2(\tilde{x},\tilde{z})\, dS \qquad \text{for heave,} \end{aligned}$$

from (III.32) and (II.26),

$$\text{(III.35)} \qquad \begin{aligned} I_x &= \frac{1}{a}\iint_{\tilde{S}} \tilde{z}\, [\, \tilde{z} - f_r^4(\tilde{x},\tilde{z})]\, dS, \\ H_x &= -\frac{1}{a}\iint_{\tilde{S}} \tilde{z}\, f_i^4(\tilde{x},\tilde{z})\, dS \qquad \text{for roll,} \end{aligned}$$

and from (III.33) and (II.27),

$$\text{(III.36)} \qquad \begin{aligned} I_z &= -\frac{1}{a}\iint_{\tilde{S}} \tilde{x}\, [\, \tilde{x} - f_r^6(\tilde{x},\tilde{z})]\, dS, \\ H_z &= \frac{1}{a}\iint_{\tilde{S}} \tilde{x}\, f_i^6(\tilde{x},\tilde{z})\, dS \qquad \text{for pitch.} \end{aligned}$$

IV. Development of Asymptotic Solution

The inspection of Green's function given by (III.16) shows that it is a function of a parameter a, hence the solution of the integral equation (III.6) must depend upon a. We develop asymptotic solution for small a in this section prior to treating a numerical procedure. The asymptotic development for the solution of two dimensional problem is presented in [6].

The functions appearing in (III.16) have expansions of the following form,

$$J_o(a\tilde{r}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{a\tilde{r}}{2}\right)^{2m} = \sum_{m=0}^{\infty} A_m(\tilde{r})\, a^{2m},$$

$$S_o(a\tilde{r}) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{[1,3,..(2m+1)]^2} (a\tilde{r})^{2m+1} = \frac{2}{\pi} \sum_{m=0}^{\infty} B_m(\tilde{r})\, a^{2m+1}$$

(IV.1)

$$Y_o(a\tilde{r}) = \frac{2}{\pi} \left(\log \frac{a\tilde{r}}{2} + \gamma\right) J_o(a\tilde{r}) + \sum_{m=1}^{\infty} (-1)^{m+1} \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) \frac{1}{(m!)^2} \left(\frac{a\tilde{r}}{2}\right)^{2m}$$

$$= \frac{2}{\pi} \left[\log a \sum_{m=0}^{\infty} A_m(\tilde{r})\, a^{2m} + \sum_{m=0}^{\infty} C_m(\tilde{r})\, a^{2m}\right].$$

Hence (III.16) can be expressed as,

$$G(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta}:a) = \frac{2}{\tilde{r}} - 2\Big[\log a \sum_{m=0}^{\infty} A_m(\tilde{r})\, a^{2m+1} + \sum_{m=0}^{\infty} C_m(\tilde{r})\, a^{2m+1} \tag{IV.2}$$

$$+ \sum_{m=0}^{\infty} B_m(\tilde{r})\, a^{2m+2} - i\pi \sum_{m=0}^{\infty} A_m(\tilde{r})\, a^{2m+1}\Big],$$

where,

$$A_o = 1, \quad A_1 = -\frac{\tilde{r}^2}{2^2}, \quad A_2 = \frac{\tilde{r}^4}{2^2,4^2}, \quad \ldots,$$

$$B_o = \tilde{r}, \quad B_1 = -\frac{\tilde{r}^3}{1,3}, \quad B_2 = \frac{\tilde{r}^5}{1,3,5}, \quad \ldots,$$

$$C_o = \gamma + \log \frac{\tilde{r}}{2}, \quad C_1 = -\left(\gamma + \log \frac{\tilde{r}}{2} - 1\right) \frac{\tilde{r}^2}{2^2}, \quad \ldots .$$

Rearranging power series we may write,

$$\text{(IV.3)} \qquad aG(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta}:a) = 2\sum_{n=0}^{\infty} \alpha_n a^{n+1} + 2\sum_{n=0}^{\infty} \beta_n a^{2n+2} \log a$$

with new coefficients,

$$\alpha_o = \frac{1}{\tilde{r}}, \quad \alpha_1 = i\pi - \gamma - \log\frac{\tilde{r}}{2}, \quad \alpha_2 = -\tilde{r}, \quad \ldots,$$

$$\beta_o = -1, \quad \beta_1 = 0, \quad \beta_2 = -\frac{\tilde{r}^2}{2}, \quad \beta_3 = 0, \quad \ldots .$$

Now we proceed to develop the asymptotic solution of the integral equation in the form,

$$\text{(IV.4)} \qquad f(x,z) \backsim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}\, a^i\, (a \log a)^j .$$

Note that the power product $a^i(a \log a)^j$ can be always ordered as to their rate of vanishing as a tends to zero, that is,

$$\lim \frac{a^{i'}(a \log a)^{j'}}{a^i(a \log a)^j} = 0 \quad \text{if} \quad i'+j' > i+j, \text{ or } i'+j' = i+j \text{ and } j > j'.$$

Substituting (IV.3) and (IV.4) in the integral term of (III.6) we obtain,

$$\text{(IV.5)} \qquad \frac{a}{4\pi}\iint_{\tilde{s}} f(\tilde{\xi},\tilde{\zeta})\, G(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\zeta}:a)\, d\tilde{\xi}\, d\tilde{\zeta}$$

$$= \frac{1}{2\pi}\iint_{\tilde{s}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} [\alpha_n f_{ij} a^{n+1+i+j}(\log a)^j + \beta_n f_{ij} a^{n+2+i+j}(\log a)^{j+1}]\, d\tilde{\xi} d\tilde{\zeta}$$

$$= \frac{1}{2\pi}\iint_{\tilde{s}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{pq}\, a^{p+q}(\log a)^q\, d\tilde{\xi} d\tilde{\zeta} .$$

Here we observe that indices are related as,

$$n+1+i+j = p+q, \qquad \text{and} \qquad j = q,$$

$$n+2+i+j = p+q, \qquad \text{and } j + 1 = q,$$

therefore in both cases $n = p-1-i$ so that we find,

$$A_{pq} = \sum_{i=0}^{p-1} \alpha_{p-1-i} f_{i\,q} + \sum_{i=0}^{p-1} \beta_{p-1-i} f_{i,q-1} \tag{IV.6}$$

Writing the right hand side of (III.6) in a power series,

$$h(\tilde{x},\tilde{z};a) = \sum_{n=0}^{\infty} h_n(\tilde{x},\tilde{z})\, a^n ,$$

we can express the integral equation in power series as,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} f_{pq} a^{p+q}(\log a)^q + \frac{1}{2\pi}\iint_{\tilde{s}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{pq} a^{p+q}(\log a)^q = \sum_{p=0}^{\infty} h_p a^p \quad \text{on } \tilde{S}. \tag{IV.7}$$

If $q = 0$, (IV.7) yields,

$$f_{po} = h_p - \frac{1}{2\pi}\iint_{\tilde{s}} \sum_{i=0}^{p-1} \alpha_{p-1-i} f_{io}\, d\tilde{\xi}\, d\tilde{\zeta} , \tag{IV.8}$$

and if $q \neq 0$,

$$f_{pq} = -\frac{1}{2\pi}\iint_{\tilde{s}} \sum_{i=0}^{p-1} (\alpha_{p-1-i} f_{iq} + \beta_{p-1-i} f_{iq-1})\, d\tilde{\xi}\, d\tilde{\zeta} . \tag{IV.9}$$

These expressions are recursion formulas which permit the determination of the coefficients $f_{ij}(\tilde{x},\tilde{z})$ in (IV.4) by means of iteration. For reference, we write down the first few terms, from (IV.8)

$$f_{oo} = h_o ,$$

$$f_{1o} = h_1 - \frac{1}{2\pi}\iint_{\tilde{s}} \alpha_o f_{oo}\, d\tilde{\zeta} d\tilde{\xi} = h_1 - \frac{1}{2\pi}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}}\, d\tilde{\zeta} d\tilde{\xi} ,$$

$$\begin{aligned} f_{2o} &= h_2 - \frac{1}{2\pi}\iint_{\tilde{s}} (\alpha_1 f_{oo} + \alpha_o f_{1o})\, d\tilde{\xi} d\tilde{\zeta} \\ &= h_2 - \frac{1}{2\pi}\iint_{\tilde{s}} \left[(i\pi - \gamma - \log\frac{\tilde{r}}{2}) h + \frac{1}{\tilde{r}} \left(h_1 - \frac{1}{2\pi}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}}\, d\tilde{\zeta} d\tilde{\zeta}\right)\right] d\tilde{\xi} d\tilde{\zeta} , \end{aligned} \tag{IV.10}$$

$$f_{3o} = h_3 - \frac{1}{2\pi}\iint_{\tilde{s}} (\alpha_2 f_{oo} + \alpha_1 f_{1o} + \alpha_o f_{2o})\, d\tilde{\xi} d\tilde{\zeta}$$

$$= h_3 - \frac{1}{2\pi}\iint_{\tilde{S}}\Big[(-\tilde{r})h_o+(i\pi-\gamma-\log\frac{\tilde{r}}{2})(h_1-\frac{1}{2\pi}\iint_{\tilde{S}}\frac{h_o}{\tilde{r}}d\xi d\zeta)$$

$$+\frac{1}{\tilde{r}}\Big\{h_2-\frac{1}{2\pi}\iint_{\tilde{S}}[(i\pi-\gamma-\log\frac{\tilde{r}}{2})h_o+\frac{1}{r}(h_1$$

$$-\frac{1}{2\pi}\iint_{\tilde{S}}\frac{h_o}{\tilde{r}}d\xi d\zeta)]d\xi d\zeta\Big\}\Big]d\xi d\zeta\,.$$

We find from (IV.9) $f_{oq} = 0$ for all q, and

$$f_{11} = -\frac{1}{2\pi}\iint_{\tilde{S}}(\alpha_o f_{o1}+\beta_o f_{oo})\,d\xi d\zeta = \frac{1}{2\pi}\iint_{\tilde{S}} h_o\,d\xi d\zeta\,,$$

$$\text{(IV.11)}\qquad f_{12} = -\frac{1}{2\pi}\iint_{\tilde{S}}(\alpha_1 f_{o2}+\beta_o f_{o1})\,d\xi d\zeta = 0,$$

$$f_{21} = -\frac{1}{2\pi}\iint_{\tilde{S}}(\alpha_1 f_{o1}+\alpha_o f_{11}+\beta_1 f_{oo}+\beta_o f_{1o})\,d\xi d\zeta$$

$$= -\frac{1}{2\pi}\iint_{\tilde{S}}[\frac{1}{\tilde{r}}(\frac{1}{2\tilde{r}}\iint_{\tilde{S}} h_o d\xi d\zeta)-(h_1-\frac{1}{2\pi}\iint_{\tilde{S}}\frac{h_o}{\tilde{r}}d\xi d\zeta)]\,d\xi d\zeta\,.$$

The above process ultimately leads to (IV.4). We say $f(\tilde{x},\tilde{z}:a)$ has an estimate of degree (i,j) if,

$$f(\tilde{x},\tilde{z}:a) = P(a,\ a\log a) + o[a^i(a\log a)^j],$$

where P is a polynominal of degree (i,j). From (IV.3) we have,

$$aG = -2\,a^2\log a + o(a^2\log a),$$

hence we see that the product of aG with a polynominal of degree (i,j) is a polynominal of degree (i+1,j+1) plus terms $o[a^{i+1}(a\log a)^{j+1}]$. Suppose then that we have shown $f(\tilde{x},\tilde{z}:a)$ to have an estimate of degree (i,j). Substituting this estimate in the integral in (IV.7) we obtain a polynominal of degree (i+1,j+1) plus terms of $o[a^{i+1}(a\log a)^{j+1}]$, with coefficients determined by the known quantities $f_{oo}, \ldots, f_{ij}$. Right hand side of (IV.7) has estimates of degree (i,0) for all i, hence (IV.7) yields for $f(\tilde{x},\tilde{z}:a)$ an estimate of degree (i+1,j+1). Then substituting the series (IV.4) and (IV.3) in the integral representation (III.1) it can be seen that $V(\tilde{x},\tilde{y},\tilde{z}:a)$ has the form,

$$aV(\tilde{x},\tilde{y},\tilde{z}:a) \backsim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} V_i(\tilde{x},\tilde{y},\tilde{z})\ a^i(a\ \log a)^j\ .$$

Now retaining the terms with h_o in the coefficients $f_{ij}(\tilde{x},\tilde{z})$ we write,

$$\text{(IV.12)} \qquad f(\tilde{x},\tilde{z}:a) \backsim f_{oo} + af_{1o} + a(a\ \log a)f_{11} + a^2 f_{2o} + a^2(a\ \log a)f_{21} + a^3 f_{3o},$$

where

$$f_{oo} = h_o\ ,$$

$$f_{1o} \approx -\frac{1}{2\pi}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}}\ d\tilde{\xi}\ d\tilde{\zeta}\ ,$$

$$f_{11} = \frac{1}{2\pi}\iint_{\tilde{s}} h_o\ d\tilde{\xi}\ d\tilde{\zeta}\ ,$$

$$\text{(IV.13)} \qquad f_{2o} \approx -\frac{1}{2\pi}\iint_{\tilde{s}} [(i\pi-\gamma-\log\frac{\tilde{r}}{2})h_o - \frac{1}{2\pi\tilde{r}}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}} d\tilde{\xi}d\tilde{\zeta}\ )]d\tilde{\xi}d\tilde{\zeta}\ ,$$

$$f_{21} \approx -\frac{1}{2\pi}\iint_{\tilde{s}} (\frac{1}{2\pi\tilde{r}}\iint_{\tilde{s}} h_o\ d\tilde{\xi}d\tilde{\zeta} + \frac{1}{2\pi}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}} d\tilde{\xi}d\tilde{\zeta}\ )d\tilde{\xi}d\tilde{\zeta}\ ,$$

$$f_{3o} \approx -\frac{1}{2\pi}\iint_{\tilde{s}} \Big[(-\tilde{r})h_o - (i\pi-\gamma-\log\frac{\tilde{r}}{2})\ \frac{1}{2\pi}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}}\ d\tilde{\xi}d\tilde{\zeta}$$

$$- \frac{1}{\tilde{r}}\Big\{\frac{1}{2\pi}\iint_{\tilde{s}} [(i\pi-\gamma-\log\frac{\tilde{r}}{2})h_o + \frac{1}{2\pi\tilde{r}}\iint_{\tilde{s}} \frac{h_o}{\tilde{r}} d\tilde{\xi}d\tilde{\zeta}]d\tilde{\xi}d\tilde{\zeta}\Big\}\Big]d\tilde{\xi}d\tilde{\zeta}\ .$$

According to (II.12) h_o is equal to 1 for the heave, $\tilde{z}$ for the roll, and $\tilde{x}$ for the pitch. Hence the integral $\iint_{\tilde{s}} h_o d\tilde{\xi}d\tilde{\zeta}$ becomes the water plane area in the case of heave, and vanishes in other cases.

Substituting the asymptotic solution (IV.12) in (III.34) - (III.36) we obtain the first non-vanishing term of the normalized added mass, added moment of inertia, and damping factor as,

$$M_y = -\frac{1}{a}\iint_{\tilde{s}} (\ \frac{a}{2\pi}\iint_{\tilde{s}} \frac{1}{\tilde{r}}\ d\tilde{\xi}\ d\tilde{\zeta}\ )\ d\tilde{x}d\tilde{z} = O(1),$$

$$N_y = -\frac{1}{a}\iint_{\tilde{s}} (\ \frac{a^2}{2}\iint_{\tilde{s}} d\tilde{\xi}\ d\tilde{\zeta}\ )\ d\tilde{x}d\tilde{z} = a\ \tilde{S}^2 = O(a),$$

$$
\begin{aligned}
I_x &= \frac{1}{a}\iint_{\tilde{S}} \left(\frac{a}{2\pi}\iint_{\tilde{S}} \frac{\tilde{\zeta}}{\tilde{r}}\, d\tilde{\xi}d\tilde{\zeta} \right) \tilde{z}\, d\tilde{x}d\tilde{z} = O(1), \\
H_x &= -\frac{1}{a}\iint_{\tilde{S}} \left[\frac{a^3}{2\pi}\iint_{\tilde{S}} \left(\frac{1}{2}\iint_{\tilde{S}} \frac{\tilde{\zeta}}{\tilde{r}}\, d\tilde{\xi}d\tilde{\zeta} \right) d\tilde{\xi}d\tilde{\zeta} \right] \tilde{z}\, d\tilde{x}d\tilde{z} = O(a^2), \\
I_z &= -\frac{1}{a}\iint_{\tilde{S}} \left(\frac{a}{2\pi}\iint_{\tilde{S}} \frac{\tilde{\xi}}{\tilde{r}}\, d\tilde{\xi}d\tilde{\zeta} \right) \tilde{x}\, d\tilde{x}d\tilde{z} = O(1), \\
H_z &= \frac{1}{a}\iint_{\tilde{S}} \left[\frac{a^3}{2\pi}\iint_{\tilde{S}} \left(\frac{1}{2}\iint_{\tilde{S}} \frac{\tilde{\xi}}{\tilde{r}}\, d\tilde{\xi}d\tilde{\zeta} \right) d\tilde{\xi}d\tilde{\zeta} \right] \tilde{x}\, d\tilde{x}d\tilde{z} = O(a^2).
\end{aligned}
\tag{IV.14}
$$

Note that N_y is equal to the product of the parameter a and the square of normalized water plane area S. As the parameter a tends to zero, N_y, H_x, and H_z vanish while M_y, I_x, and I_z approach non-zero constants. This is to be compared with the situation in two dimensional case where N_y becomes a constant while M_y tends to infinity as shown in [5] and [6]. These estimates indicate that the strip method shown in [5] is not expected to yield accurate results for low frequency.

V. Numerical Procedure

We suppose the surface of a flat ship to be an ellipse given by $(\bar{x}/\bar{a})^2 + (\bar{z}/\bar{b})^2 = 1$, or in normalized co-ordinates $\tilde{x}^2 + (\tilde{z}/\tilde{b})^2 = 1$ with $\tilde{b} = \bar{b}/\bar{a}$, and develop a numerical procedure by which the value of unknown function $f(\tilde{x},\tilde{z})$ in the integral equation (III.6) can be determined approximately. We replace the equation (III.6) by a set of linear equations relating the values of $f(\tilde{x},\tilde{z})$ at chosen pivotal points on the elliptic surface. Then the surface integrals in the linear equations are evaluated by Simpson's rule given by,

$$
\int_c^d \int_a^b f(\tilde{x},\tilde{z})\,d\tilde{x}d\tilde{z} \approx \int_c^d \left[h \sum_{l=0}^{p} C_l f(\tilde{x}_l,\tilde{z})\right] d\tilde{z} \approx k \sum_{m=0}^{q} D_m \left[h \sum_{l=0}^{p} C_l f(\tilde{x}_l,\tilde{z}_m)\right]
\tag{V.1}
$$

for a continuous function $f(\tilde{x},\tilde{z})$. Here $h = (b-a)/p$, $k = (d-c)/q$, and C_l takes the values $1/3$ for $l=0$ or p, $4/3$ for $l=2n$, and $2/3$ for $l=2n+1$, respectively. D_m takes the values $1/3$ for $m=0$ or q, $4/3$ for $m=2n$, and $2/3$ for $m=2n+1$, respectively.

Substitution of (III.16) in (III.6) yields,

$$2[f_r(\tilde{x},\tilde{z})+if_i(\tilde{x},\tilde{z})]+\frac{a}{2\pi}\iint_{\tilde{S}}[f_r(\tilde{\xi},\tilde{\zeta})+if_i(\tilde{\xi},\tilde{\zeta})]\left\{\frac{2}{\tilde{r}}-\pi a[Y_o(a\tilde{r})+S_o(a\tilde{r})-i2J_o(a\tilde{r})]\right\}d\tilde{\xi}d\tilde{\zeta}$$

$$= 2h(\tilde{x},\tilde{z}) \qquad \text{on} \quad \tilde{y} = S(\tilde{x},\tilde{z}).$$

If we write the real and imaginary parts separately, the following pair of equations will result.

$$2f_r(\tilde{x},\tilde{z})+\frac{a}{\pi}\iint_{\tilde{S}}\left\{f_r(\tilde{\xi},\tilde{\zeta})[\frac{1}{\tilde{r}}-a\log(a\tilde{r})-\frac{\pi a}{2}R(a\tilde{r})]-f_i(\tilde{\xi},\tilde{\zeta})\pi aJ_o(a\tilde{r})\right\}d\tilde{\xi}d\tilde{\zeta}=2h(\tilde{x},\tilde{z}),$$

(V.2)

$$2f_i(\tilde{x},\tilde{z})+\frac{a}{\pi}\iint_{\tilde{S}}\left\{f_r(\tilde{\xi},\tilde{\zeta})\pi aJ_o(a\tilde{r})+f_i(\tilde{\xi},\tilde{\zeta})[\frac{1}{\tilde{r}}-a\log(a\tilde{r})-\frac{\pi a}{2}R(a\tilde{r})]\right\}d\tilde{\xi}d\tilde{\zeta}=0,$$

where $\qquad R(a\tilde{r}) = Y_o(a\tilde{r}) + S_o(a\tilde{r}) - \frac{2}{\pi}\log(a\tilde{r}).$

We now establish a lattice on the elliptic surface $\tilde{S}$ by dividing the long axis into eight equal intervals h, and the vertical ordinates parallel to the short axis into four equal intervals $k(\tilde{x})$, that is, $h = 1/4$ and $k(\tilde{x}) = b\sqrt{1-\tilde{x}^2}/2$. In the course of additional computations for improving numerical results these intervals are bisected to yield a finer grid. However we only present here the procedure for original lattice. In this instance, each pivotal point can be identified by the coordinates $\tilde{x}_i$ and $\tilde{z}_j$ where $\tilde{x}_i=(i-4)/4$ $i=0,1,\ldots 8$, and $\tilde{z}_j(\tilde{x}_i)=$ $\tilde{b}\,\frac{j-2}{2}\sqrt{1-\tilde{x}_i^2}$ $j=0,1,\ldots 4$. To determine the values of $f(\tilde{x},\tilde{z})$, we consider the equation (V.2) only at thirteen pivotal points contained in one quadrant. [Note that for the fine lattice, the pivotal points contained in one quadrant are forty-one.] When the values of $f(\tilde{x},\tilde{z})$ at these points are known, the symmetry or anti-symmetry properties of the function will enable us to determine the values at the rest of pivotal points.

As $\tilde{r}$ denotes the distance from a fixed pivotal point $(\tilde{x}_i,\tilde{z}_j)$ to any pivotal point $(\tilde{\xi},\tilde{\zeta}) = (\tilde{x}_l,\tilde{z}_m)$ where $l=0,1,\ldots 8$, and $m=0,1,\ldots 4$, we find the integrands,

$$f(\xi,\zeta)/\sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\xi)^2} \quad \text{and} \quad f(\xi,\zeta)\ \log a \sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2}$$

possess a singularity at $(\tilde{x}_i,\tilde{z}_j)=(\tilde{x}_l,\tilde{z}_m)$. Nevertheless, the singular integrals associated with these integrands do exist. We will presently show how these integrals can be evaluated.

(i) Treatment of the integral

$$I_1(\tilde{x}_i,\tilde{z}_j)=\iint_{\tilde{S}} \frac{f(\xi,\zeta)}{\sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2}}\, d\xi d\zeta .$$

We choose the shortest distance r_o between any two neighboring pivotal points and draw a circular region about the fixed pivotal point. If the integral $I_1(\tilde{x}_i,\tilde{z}_j)$ is evaluated over the region $\tilde{S}-\delta$, and the excluded circular region δ separately,

$$\text{(V.3)} \qquad I_1(\tilde{x}_i,\tilde{z}_j)\approx\iint_{\tilde{S}-\delta} \frac{f(\tilde{\xi},\tilde{\zeta})}{\sqrt{(\tilde{x}_i-\tilde{\xi})^2+(\tilde{z}_j-\tilde{\zeta})^2}}\, d\tilde{\xi}d\tilde{\zeta} + f(\tilde{x}_i,\tilde{z}_j)\iint_{\delta} \frac{d\tilde{\xi}d\tilde{\zeta}}{\sqrt{(\tilde{x}_i-\tilde{\xi})^2+(\tilde{z}_j-\tilde{\zeta})^2}} ,$$

where the function $f(\tilde{x},\tilde{z})$ is regarded as a constant over the region δ. Since the integral over the circular region becomes,

$$\int_0^{2\pi}\int_0^{r_o} \frac{1}{\tilde{r}}\, d\tilde{r}\ \tilde{r}d\theta = 2\pi r_o ,$$

we obtain,

$$\text{(V.4)} \qquad I_1(\tilde{x}_i,\tilde{z}_j) \approx 2\pi r_o f(\tilde{x}_i,\tilde{z}_j) + \iint_{\tilde{S}-\delta} \frac{f(\tilde{\xi},\tilde{\zeta})}{\sqrt{(\tilde{x}_i-\tilde{\xi})^2+(\tilde{z}_j-\tilde{\zeta})^2}}\, d\tilde{\xi}d\tilde{\zeta} .$$

Now suppose the integral $I_1(\tilde{x}_i,\tilde{z}_j)$ is evaluated by (V.1) assigning a fictitious value r_o for $\tilde{r}$ at the singularity,

$$\iint_{\tilde{S}-\delta} \frac{f(\tilde{\xi},\tilde{\zeta})}{\sqrt{(\tilde{x}_i-\tilde{\xi})^2+(\tilde{z}_j-\tilde{\zeta})^2}}\, d\tilde{\xi}d\tilde{\zeta} + \frac{f(\tilde{x}_i,\tilde{z}_j)}{r_o}\iint_{\delta} d\tilde{\xi}d\tilde{\zeta}$$

$$\approx \sum_{l=0}^{8}\sum_{m=0}^{4} C_l h\, D_m k(\tilde{x}_l) \frac{f(\tilde{x}_l,\tilde{z}_m)}{\sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}} ,$$

where the notation $\sum\sum_{\varrho}$ denotes the double summation with $\tilde{r} = r_o$, inside the circle. C_l and D_m are the coefficients of Simpson's rule in the $\tilde{x}$- and $\tilde{z}$-directions, respectively.
Hence we find,

$$\text{(V.5)}\qquad \iint_{\tilde{S}-\delta} \frac{f(\xi,\zeta)}{\sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2}}\, d\xi d\zeta \approx -\pi r_o f(\tilde{x}_i,\tilde{z}_j)$$

$$+ \sum_{l=0}^{8}\sum_{m=0}^{4}{}_{\varrho}\; C_l h D_m k(\tilde{x}_l) \frac{f(\tilde{x}_l,\tilde{z}_m)}{\sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}} \;.$$

Substitution of (V.5) in (V.4) yields,

$$\text{(V.6)}\qquad I_1(x_i,z_j) \approx \pi r_o f(\tilde{x}_i,\tilde{z}_j) + \sum_{l=0}^{8}\sum_{m=0}^{4}{}_{\varrho}\; C_l h D_m k(\tilde{x}_l) \frac{f(\tilde{x}_l,\tilde{z}_m)}{\sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}} \;.$$

For pivotal points on the boundary, the correction term $\pi r_o f(\tilde{x}_i,\tilde{z}_j)$ will be approximated by one half because the circular region drawn with the radius r_o about such a pivotal point does not make a full circle as the region will be sliced off by the boundary of the ellipse.

(ii) Treatment of the integral

$$I_2(\tilde{x}_i,\tilde{z}_j) = \iint_{\tilde{S}} f(\xi,\zeta) \log a \sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2} \;.$$

If the integral $I_2(\tilde{x}_i,\tilde{z}_j)$ is evaluated separately over the region $\tilde{S}-\delta$ and δ,

$$\text{(V.7)}\qquad I_2(\tilde{x}_i,\tilde{z}_j) \approx \iint_{\tilde{S}-\delta} f(\xi,\zeta) \log a \sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2}\, d\xi d\zeta$$

$$+ f(\tilde{x}_i,\tilde{z}_j) \iint_{\delta} \log a \sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2}\, d\xi d\zeta \;,$$

where $f(\tilde{x},\tilde{z})$ is again regarded as constant over the region δ.
Since the integral over the circular region becomes,

$$\int_0^{2\pi}\int_0^{r_o} \log a\tilde{r}\, d\tilde{r}\, \tilde{r} d\theta = \pi r_o^2(\log a r_o - \tfrac{1}{2}),$$

we have,

$$(\text{V.8}) \qquad I_2(\tilde{x}_i,\tilde{z}_j) \approx \pi r_o^2(\log a r_o - \tfrac{1}{2}) f(\tilde{x}_i,\tilde{z}_j) + \iint_{\tilde{S}-\delta} f(\xi,\zeta) \log a \sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\zeta)^2}\, d\xi d\zeta.$$

Assigning a fictitious value r_o for $\tilde{r}$ at the singularity, and evaluating $I_2(\tilde{x}_i,\tilde{z}_j)$ by (V.1) we obtain,

$$\iint_{\tilde{S}-\delta} f(\xi,\tilde{\zeta}) \log a \sqrt{(\tilde{x}_i-\xi)^2+(\tilde{z}_j-\tilde{\zeta})^2}\, d\xi d\zeta + f(\tilde{x}_i,\tilde{z}_j) \log a r_o \iint_{\delta} d\xi d\tilde{\zeta}$$

$$\approx \sum_{l=0}^{8} \sum_{m=0}^{4} C_l h D_m k(\tilde{x}_l) \log a \sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}\, f(\tilde{x}_l,\tilde{z}_m).$$

Therefore,

$$(\text{V.9}) \qquad \iint_{\tilde{S}-\delta} f(\tilde{\xi},\tilde{\zeta}) \log a \sqrt{(\tilde{x}_i-\tilde{\xi})^2+(\tilde{z}_j-\tilde{\zeta})^2}\, d\tilde{\xi} d\tilde{\zeta} \approx -\pi r_o^2 \log a r_o f(\tilde{x}_i,\tilde{z}_j)$$

$$+ \sum_{l=0}^{8} \sum_{m=0}^{4} C_l h D_m k(\tilde{x}_l) \log a \sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}\, f(\tilde{x}_l,\tilde{z}_m).$$

Substitution of (V.9) in (V.8) yields,

$$(\text{V.10}) \qquad I_2(\tilde{x}_i,\tilde{z}_j) \approx -\frac{\pi}{2} r_o^2 f(\tilde{x}_i,\tilde{z}_j)$$

$$+ \sum_{l=0}^{8} \sum_{m=0}^{4} C_l h D_m k(\tilde{x}_l) \log a \sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}\, f(\tilde{x}_l,\tilde{z}_m).$$

Now the application of Simpson's rule (V.1) together with the singular integral formulae (V.6) and (V.10) enable us to reduce the set of integral equation (V.2) to twenty-six (for the original lattice or eighty-two for the fine lattice) linear equations relating the values of $f_r(\tilde{x},\tilde{z})$ and $f_i(\tilde{x},\tilde{z})$ at chosen pivotal points.

For convenience we use the following notations,

$$(\text{V.11}) \qquad \begin{aligned} f^{ij} &= f(\tilde{x}_i,\tilde{z}_j), \quad f^{lm} = f(\tilde{x}_l,\tilde{z}_m), \quad \tilde{r}_{lm} = \sqrt{(\tilde{x}_i-\tilde{x}_l)^2+(\tilde{z}_j-\tilde{z}_m)^2}, \\ K_{lm} &= C_l h\, D_m k(\tilde{x}_l), \quad \text{and} \quad h^{ij} = h(\tilde{x}_i,\tilde{z}_j). \end{aligned}$$

We write (V.2) as,

$$[2 + n a r_o(1 - \frac{a r_o}{2})] f_r^{ij} + \frac{a}{\pi} \sum_{l=0}^{8} \sum_{m=0}^{4} K_{lm} [\frac{1}{\tilde{r}_{lm}} - a \log a\tilde{r}_{lm} - \frac{\pi a}{2} R(a\tilde{r}_{lm})] f_r^{lm}$$

$$- a^2 \sum_{l=0}^{8} \sum_{m=0}^{4} K_{lm} J_o(a\tilde{r}_{lm}) f_i^{lm} = 2h^{ij} ,$$

(V.12)

$$a^2 \sum_{l=0}^{8} \sum_{m=0}^{4} K_{lm} J_o(a\tilde{r}_{lm}) f_r^{lm} + [2 + n a r_o(1 - \frac{a r_o}{2})] f_i^{ij}$$

$$+ \frac{a}{\pi} \sum_{l=0}^{8} \sum_{m=0}^{4} K_{lm} [\frac{1}{\tilde{r}_{lm}} - a \log a\tilde{r}_{lm} - \frac{\pi a}{2} R(a\tilde{r}_{lm})] f_i^{lm} = 0,$$

where n is a function of (i,j) which takes the value of 1/2 or 1 . Note that among the coefficients of Simpson's rule the relation, $K_{l,m} = K_{9-l,m} = K_{l,5-m} = K_{9-l,5-m}$, holds. We further write,

$$H_1^{lm} = \frac{1}{\tilde{r}_{lm}} - a \log a\tilde{r}_{lm} - \frac{\pi a}{2} R(a\tilde{r}_{lm}),$$

(V.13)

$$H_2^{lm} = J_o(a\tilde{r}_{lm}), \text{ and } C = a r_o(1 - \frac{a r_o}{2}).$$

Next we shall investigate the charecteristics of linear equations associated with the heave, roll and pitch, respectively:

(i) The case of heave.

Since the function $f(\tilde{x},\tilde{z})$ satisfies the symmetry relations,

$$f^{l,m} = f^{9-l,m}, \text{ and } f^{l,m} = f^{l,5-m} \quad \text{for heave,}$$

the double sums appearing in (V.12) can be expressed as,

$$\text{(V.14)} \quad \sum_{l=0}^{8} \sum_{m=0}^{4} K_{lm} H^{lm} f^{lm} = \sum_{l=0}^{4} \sum_{m=0}^{2} K_{lm} (H^{l,m} + H^{l,5-m} + H^{9-l,m} + H^{9-l,5-m}) f^{lm}.$$

From (V.13) and (V.14) the linear equations (V.12) for heave becomes,

$$(2+nC)f_r^{ij} + \frac{a}{\pi}\sum_{l=0}^{4}\sum_{m=0}^{2}{}^{\prime} K_{lm}(H_1^{l,m}+H_1^{l,5-m}+H_1^{9-l,m}+H_1^{9-l,5-m})f_r^{lm}$$

$$-a^2\sum_{l=0}^{4}\sum_{m=0}^{2} K_{lm}(H_2^{l,m}+H_2^{l,5-m}+H_2^{9-l,m}+H_2^{9-l,5-m})f_i^{lm} = 2,$$

(V.15)

$$(2+nC)f_i^{ij} + \frac{a}{\pi}\sum_{l=0}^{4}\sum_{m=0}^{2}{}^{\prime} K_{lm}(H_1^{l,m}+H_1^{l,5-m}+H_1^{9-l,m}+H_1^{9-l,5-m})f_i^{lm}$$

$$-a^2\sum_{l=0}^{4}\sum_{m=0}^{2} K_{lm}(H_2^{l,m}+H_2^{l,5-m}+H_2^{9-l,m}+H_2^{9-l,5-m})f_r^{lm} = 0.$$

The coefficient matrix of (V.15) is anti-symmetric, and its elements in the first and the fourteenth columns and row will vanish except the main diagonal elements because in the process of the double integration by (V.1) the function at the both ends of the major axis are not taken into account.

(ii) The case of roll.

For roll the function $f(\tilde{x},\tilde{z})$ satisfies the symmetry relation,

$f^{l,m}= f^{9-l,m}$, and the anti-symmetry relation $f^{l,m}=f^{9-l,m}=-f^{l,5-m}=-f^{9-l,5-m}$, therefore the double sums appearing in (V.12) for the roll becomes,

(V.16) $$\sum_{l=0}^{8}\sum_{m=0}^{4} K_{lm}H^{lm}f^{lm} = \sum_{l=0}^{4}\sum_{m=0}^{2} K_{lm}(H^{l,m}-H^{l,5-m}+H^{9-l,m}-H^{9-l,5-m})f^{lm}.$$

From (V.13) and (V.16) the linear equations (V.12) for the roll becomes,

$$(2+nC)f_r^{ij} + \frac{a}{\pi}\sum_{l=0}^{4}\sum_{m=0}^{2}{}^{\prime} K_{lm}(H_1^{l,m}-H_1^{l,5-m}+H_1^{9-l,m}-H_1^{9-l,5-m})f_r^{lm}$$

$$-a^2\sum_{l=0}^{4}\sum_{m=0}^{2} K_{lm}(H_2^{l,m}-H_2^{l,5-m}+H_2^{9-l,m}-H_2^{9-l,5-m})f_i^{lm}=-2z,$$

(V.17)

$$(2+nC)f_i^{ij} + \frac{a}{\pi}\sum_{l=0}^{4}\sum_{m=0}^{2}{}^{\prime} K_{lm}(H_1^{l,m}-H_1^{l,5-m}+H_1^{9-l,m}-H_1^{9-l,5-m})f_i^{lm}$$

$$-a^2\sum_{l=0}^{4}\sum_{m=0}^{2}K_{lm}(H_2^{l,m}-H_2^{l,5-m}+H_2^{9-l,m}-H_2^{9-l,5-m})f_r^{lm}=0.$$

Here inspection of (V.16) shows that,

$$H^{l,m}-H^{l,5-m}+H^{9-l,m}-H^{9-l,5-m}=0 \qquad \text{for } m=2$$

(V.18)

$$\sum_{l=0}^{4}\sum_{m=0}^{2}(H^{l,m}-H^{l,5-m}+H^{9-l,m}-H^{9-l,5-m})=0 \quad \text{for} \quad j=2.$$

Therefore the coefficient matrix of (V.17) is also anti-symmetric and in addition to vanishing first and fourteenth columns and rows, from (V.18) it has zeroes in the $(4+3m)$th and $(17+3m)$th columns and rows except the main diagonal elements, m here assumes the values 0,1,2,3.

(iii) The case of pitch.

For pitch the function $f(\tilde{x},\tilde{z})$ has the symmetry property, $f^{l,m}=f^{l,5-m}$, and the anti-symmetry property, $f^{l,m}=f^{l,5-m}=-f^{9-l,m}=-f^{9-l,5-m}$, therefore the double sums appearing in (V.12) for the pitch becomes,

$$\text{(V.19)}\quad \sum_{l=0}^{8}\sum_{m=0}^{4}K_{lm}H^{lm}f^{lm}=\sum_{l=0}^{4}\sum_{m=0}^{2}K_{lm}(H^{l,m}+H^{l,5-m}-H^{9-l,m}-H^{9-l,5-m})f^{lm}.$$

From (V.13) and (V.19) we write the linear equations (V.12) for pitch as,

$$(2+nC)f_r^{ij}+\frac{a}{\pi}\sum_{l=0}^{4}\sum_{m=0}^{2}K_{lm}(H_1^{l,m}+H_1^{l,5-m}-H_1^{9-l,m}-H_1^{9-l,5-m})f_r^{lm}$$

$$-a^2\sum_{l=0}^{4}\sum_{m=0}^{2}K_{lm}(H_2^{l,m}+H_2^{l,5-m}-H_2^{9-l,m}-H_2^{9-l,5-m})f_i^{lm}=-2x,$$

(V.20)

$$(2+nC)f_i^{ij}+\frac{a}{\pi}\sum_{l=0}^{4}\sum_{m=0}^{2}K_{lm}(H_1^{l,m}+H_1^{l,5-m}-H_1^{9-l,m}-H_1^{9-l,5-m})f_i^{lm}$$

$$-a^2\sum_{l=0}^{4}\sum_{m=0}^{2}K_{lm}(H_2^{l,m}+H_2^{l,5-m}-H_2^{9-l,m}-H_2^{9-l,5-m})f_r^{lm}=0.$$

Here it can be seen that,

$$H^{l,m}+H^{l,5-m}-H^{9-l,m}-H^{9-l,5-m} = 0 \qquad \text{for } l = 4,$$

(V.21)

$$\sum_{l=0}^{4}\sum_{m=0}^{2}(H^{l,m}+H^{l,5-m}-H^{9-l,m}-H^{9-l,5-m}) = 0 \qquad \text{for } i = 4.$$

Therefore the coefficient matrix of (V.20) is anti-symmetric, and from (V.21) it has vanishing elements in the (11+m)th and (24+m) th columns and rows except the main diagonal elements, m here assumes the values 0,1,2.

The integral equation (V.6) describing the forced oscillation of a flat ship is thus replaced by three sets of linear equations. For various values of the parameter $a = \sigma^2\bar{a}/g = 2\pi\bar{a}/\bar{\lambda}$, we can determine the values of $f(\tilde{x},\tilde{z})$ by solving these equations. Actual steps of the computation work for obtaining the solution consist of :

(1) Determination of the distance $\tilde{r}$ from individual pivotal point to any pivotal point on the given lattice.

(2) Calculation of functions appearing in the coefficients of the linear equations either by direct evaluation or by interpolation from the given table.

(3) Summation of the coefficients and grouping of the matrix in accordance with the type of motion.

(4) Numerical solution of linear equations either by the elimination process or iteration process.

In step (1) when the distance $\tilde{r}$ becomes zero, it is replaced by the smallest distance between two neighboring pivotal points r_0 , and in step (2) the functions, $1/\tilde{r}$ and $\log a\tilde{r}$ are evaluated directly and the function $R(a\tilde{r}) = Y_0(a\tilde{r}) + S_0(a\tilde{r}) - \frac{2}{\pi}\log a\tilde{r}$ and $J_0(a\tilde{r})$ are evaluated by means of parabolic interpolation from a pre-arranged table [Table-1]. In step (4) successive elimination of unknowns based on the algorithm of Gauss is used.

Table 1

Input for Determination of Coefficients of Linear Equation

$a\tilde{r}$	$R(a\tilde{r})$	$J_o(a\tilde{r})$	$a\tilde{r}$	$R(a\tilde{r})$	$J_o(a\tilde{r})$	$a\tilde{r}$	$R(a\tilde{r})$	$J_o(a\tilde{r})$
0.00	-0.0738	1.0000	2.25	0.7734	0.0828	4.50	-1.2107	-0.3205
0.05	-0.0401	0.9994	2.30	0.7552	0.0555	4.55	-1.2487	-0.3087
0.10	-0.0045	0.9975	2.35	0.7303	0.0288	4.60	-1.2851	-0.2961
0.15	0.0320	0.9944	2.40	0.7036	0.0025	4.65	-1.3200	-0.2830
0.20	0.0704	0.9900	2.45	0.6751	-0.0232	4.70	-1.3533	-0.2693
0.25	0.1085	0.9844	2.50	0.6448	-0.0484	4.75	-1.3850	-0.2551
0.30	0.1486	0.9776	2.55	0.6124	-0.0729	4.80	-1.4151	-0.2404
0.35	0.1881	0.9696	2.60	0.5784	-0.0968	4.85	-1.4435	-0.2253
0.40	0.2274	0.9604	2.65	0.5428	-0.1200	4.90	-1.4702	-0.2097
0.45	0.2674	0.9500	2.70	0.5054	-0.1424	4.95	-1.4951	-0.1938
0.50	0.3059	0.9385	2.75	0.4667	-0.1641	5.00	-1.5183	-0.1776
0.55	0.3551	0.9258	2.80	0.4263	-0.1850	5.05	-1.5398	-0.1611
0.60	0.3831	0.9120	2.85	0.3846	-0.2051	5.10	-1.5594	-0.1443
0.65	0.4209	0.8971	2.90	0.3415	-0.2243	5.15	-1.5772	-0.1274
0.70	0.4579	0.8812	2.95	0.2972	-0.2426	5.20	-1.5933	-0.1103
0.75	0.4944	0.8642	3.00	0.2518	-0.2601	5.25	-1.6075	-0.0931
0.80	0.5293	0.8463	3.05	0.2052	-0.2765	5.30	-1.6199	-0.0758
0.85	0.5631	0.8274	3.10	0.1576	-0.2921	5.35	-1.6306	-0.0585
0.90	0.5957	0.8075	3.15	0.1093	-0.3066	5.40	-1.6394	-0.0412
0.95	0.6271	0.7868	3.20	0.0600	-0.3202	5.45	-1.6463	-0.0240
1.00	0.6570	0.7652	3.25	0.0099	-0.3328	5.50	-1.6515	-0.0068
1.05	0.6852	0.7428	3.30	-0.0507	-0.3443	5.55	-1.6552	0.0102
1.10	0.7121	0.7196	3.35	-0.0917	-0.3548	5.60	-1.6569	0.0270
1.15	0.7372	0.6957	3.40	-0.1435	-0.3643	5.65	-1.6570	0.0436
1.20	0.7606	0.6711	3.45	-0.1955	-0.3727	5.70	-1.6555	0.0599
1.25	0.7822	0.6459	3.50	-0.2478	-0.3801	5.75	-1.6523	0.0760
1.30	0.8019	0.6201	3.55	-0.3001	-0.3865	5.80	-1.6475	0.0917
1.35	0.8197	0.5937	3.60	-0.3526	-0.3918	5.85	-1.6412	0.1071
1.40	0.8354	0.5669	3.65	-0.4052	-0.3960	5.90	-1.6334	0.1220
1.45	0.8492	0.5395	3.70	-0.4574	-0.3992	5.95	-1.6240	0.1366
1.50	0.8610	0.5118	3.75	-0.5096	-0.4014	6.00	-1.6135	0.1506
1.55	0.8706	0.4838	3.80	-0.5616	-0.4026	6.05	-1.6014	0.1642
1.60	0.8782	0.4554	3.85	-0.6131	-0.4027	6.10	-1.5880	0.1773
1.65	0.8836	0.4268	3.90	-0.6641	-0.4018	6.15	-1.5735	0.1898
1.70	0.8868	0.3980	3.95	-0.7147	-0.4000	6.20	-1.5641	0.2017
1.75	0.8879	0.3690	4.00	-0.7644	-0.3971	6.25	-1.5409	0.2131
1.80	0.8867	0.3400	4.05	-0.8137	-0.3934	6.30	-1.5230	0.2238
1.85	0.8833	0.3109	4.10	-0.8620	-0.3887	6.35	-1.5041	0.2339
1.90	0.8777	0.2818	4.15	-0.9093	-0.3831	6.40	-1.4842	0.2433
1.95	0.8700	0.2528	4.20	-0.9559	-0.3766	6.45	-1.4637	0.2521
2.00	0.8601	0.2239	4.25	-1.0014	-0.3692	6.50	-1.4421	0.2601
2.05	0.8476	0.1951	4.30	-1.0457	-0.3610	6.55	-1.4200	0.2674
2.10	0.8335	0.1666	4.35	-1.0891	-0.3520	6.60	-1.3973	0.2740
2.15	0.8170	0.1383	4.40	-1.1308	-0.3423	6.65	-1.3738	0.2799
2.20	0.7984	0.1104	4.45	-1.1715	-0.3318	6.70	-1.3501	0.2851

Table 1

Input for Determination of Coefficients of Linear Equation

$a\tilde{r}$	$R(a\tilde{r})$	$J_o(a\tilde{r})$	$a\tilde{r}$	$R(a\tilde{r})$	$J_o(a\tilde{r})$	$a\tilde{r}$	$R(a\tilde{r})$	$J_o(a\tilde{r})$
7.55	-0.9437	0.2593	9.80	-1.1787	-0.2323	12.05	-1.9763	0.0588
7.60	-0.9238	0.2516	9.85	-1.2071	-0.2366	12.10	-1.9719	0.0697
7.65	-0.9048	0.2434	9.90	-1.2350	-0.2403	12.15	-1.9663	0.0803
7.70	-0.8865	0.2346	9.95	-1.2631	-0.2434	12.20	-1.9548	0.0908
7.75	-0.8691	0.2252	10.00	-1.2915	-0.2459	12.25	-1.9520	0.1009
7.80	-0.8528	0.2154	10.05	-1.3199	-0.2478	12.30	-1.9435	0.1108
7.85	-0.8375	0.2051	10.10	-1.3484	-0.2490	12.35	-1.9339	0.1203
7.90	-0.8233	0.1944	10.15	-1.3770	-0.2496	12.40	-1.9234	0.1296
7.95	-0.8103	0.1832	10.20	-1.4055	-0.2496	12.45	-1.9121	0.1384
8.00	-0.7983	0.1717	10.25	-1.4338	-0.2490	12.50	-1.8997	0.1469
8.05	-0.7876	0.1597	10.30	-1.4620	-0.2477	12.55	-1.8866	0.1550
8.10	-0.7780	0.1475	10.35	-1.4899	-0.2458	12.60	-1.8730	0.1626
8.15	-0.7698	0.1350	10.40	-1.5176	-0.2434	12.65	-1.8584	0.1698
8.20	-0.7627	0.1222	10.45	-1.5449	-0.2403	12.70	-1.8433	0.1766
8.25	-0.7570	0.1092	10.50	-1.5718	-0.2366	12.75	-1.8274	0.1829
8.30	-0.7526	0.0960	10.55	-1.5985	-0.2324	12.80	-1.8110	0.1887
8.35	-0.7494	0.0826	10.60	-1.6242	-0.2276	12.85	-1.7940	0.1940
8.40	-0.7477	0.0692	10.65	-1.6496	-0.2223	12.90	-1.7767	0.1988
8.45	-0.7472	0.0556	10.70	-1.6743	-0.2164	12.95	-1.7588	0.2031
8.50	-0.7480	0.0419	10.75	-1.6983	-0.2101	13.00	-1.7406	0.2069
8.55	-0.7494	0.0283	10.80	-1.7216	-0.2032	13.05	-1.7221	0.2102
8.60	-0.7538	0.0146	10.85	-1.7442	-0.1959	13.10	-1.7034	0.2129
8.65	-0.7586	0.0010	10.90	-1.7660	-0.1881	13.15	-1.6843	0.2151
8.70	-0.7649	-0.0125	10.95	-1.7868	-0.1798	13.20	-1.6652	0.2167
8.75	-0.7725	-0.0259	11.00	-1.8068	-0.1712	13.25	-1.6458	0.2178
8.80	-0.7812	-0.0392	11.05	-1.8259	-0.1622	13.30	-1.6267	0.2183
8.85	-0.7914	-0.0523	11.10	-1.8440	-0.1528	13.35	-1.6118	0.2183
8.90	-0.8026	-0.0653	11.15	-1.8611	-0.1430	13.40	-1.5880	0.2177
8.95	-0.8151	-0.0779	11.20	-1.8770	-0.1330	13.45	-1.5689	0.2166
9.00	-0.8290	-0.0903	11.25	-1.8922	-0.1227	13.50	-1.5498	0.2150
9.05	-0.8439	-0.1024	11.30	-1.9060	-0.1121	13.55	-1.5311	0.2128
9.10	-0.8600	-0.1142	11.35	-1.9188	-0.1012	13.60	-1.5126	0.2101
9.15	-0.8771	-0.1257	11.40	-1.9305	-0.0902	13.65	-1.4945	0.2069
9.20	-0.8954	-0.1367	11.45	-1.9409	-0.0790	13.70	-1.4767	0.2032
9.25	-0.9145	-0.1474	11.50	-1.9503	-0.0677	13.75	-1.4593	0.1990
9.30	-0.9348	-0.1577	11.55	-1.9586	-0.0562	13.80	-1.4425	0.1943
9.35	-0.9560	-0.1674	11.60	-1.9657	-0.0446	13.85	-1.4260	0.1892
9.40	-0.9780	-0.1768	11.65	-1.9713	-0.0330	13.90	-1.4102	0.1836
9.45	-1.0008	-0.1856	11.70	-1.9761	-0.0213	13.95	-1.3951	0.1775
9.50	-1.0245	-0.1939	11.75	-1.9806	-0.0097	14.00	-1.3805	0.1711
9.55	-1.0488	-0.2017	11.80	-1.9820	0.0020	14.05	-1.3667	0.1642
9.60	-1.0739	-0.2090	11.85	-1.9826	0.0135	14.10	-1.3534	0.1570
9.65	-1.0995	-0.2157	11.90	-1.9831	0.0250	14.15	-1.3412	0.1493
9.70	-1.1257	-0.2218	11.95	-1.9821	0.0364	14.25	-1.3187	0.1331
9.75	-1.1525	-0.2273	12.00	-1.9796	0.0477	14.30	-1.3087	0.1245

VI. Discussion of Numerical Results

The computations are in terms of two parameters. The first is the ratio of the short axis to the long axis of the ellipse $\tilde{b} = \bar{b}/\bar{a}$, and the second parameter is the ratio of the half length of the ship to the wave length $a = 2\pi\bar{a}/\lambda = \sigma^2\bar{a}/g$. The values of $\tilde{b}$ chosen for the investigation were 1/8, 1/4, and 1 so that they represent a slim ellipse and a circle as limiting cases. The parameter a assumes the values $\pi/6$, $\pi/5$, $\pi/4$, $\pi/3$, $2\pi/5$, $\pi/2$, $2\pi/3$, and π.

It was shown in section III that for large argument of $a\tilde{r}$, the asymptotic form of the Green's function contains a trigonometric function, that is, $G(\tilde{x},\tilde{z},\xi,\zeta) \approx 2/\tilde{r} + i2\sqrt{2\pi a/\tilde{r}}\; e^{i(a\tilde{r}-\pi/4)}$. Hence, the kernel of the integral equation fluctuates as the frequency of oscillation increases. This implies that to assure equal accuracy we must take the grid spacing inversely proportional to the frequency. For this reason, the computations are carried out at first with the original lattice and then with the fine lattice which has the bisected grid spacings.

For each combination of $\tilde{b}$ and a, three set of linear equations (V.15), (V.17), and (V.20) were solved in order to determine the values of $f_r(\tilde{x},\tilde{z})$ and $f_i(\tilde{x},\tilde{z})$ at chosen pivotal points. Then using Simpson's rule (V.1) the normalized added mass M_y and damping factor N_y for heave were evaluated by (III.34). Similary, for roll the normalized added moment of inertia I_x and damping factor H_x were evaluated by (III.35), while for pitch the normalized added moment of inertia I_z and damping factor H_z were evaluated by (III.36). In Table 2 the values of M_y, N_y, I_x, H_x, I_z, and H_z for a circular disk, $\tilde{b}=1$, corresponding to various values of the parameter a are tabulated. In Tables 3 and 4 the values of M_y, N_y, I_x, H_x, I_z, and H_z for elliptic disks of the axes ratio 1/4, and 1/8, depending upon the parameter a are presented. In these tables, the values within the parenthesis denote the results obtained by the use of the original lattice and the other values by the use of the fine lattice. Note that for the case of elliptic disks the results obtained by the use of the original lattice are not much different from those by the use of the fine lattice. However in Table 2 it can be seen that the

Table 2

Added Mass, Added Moment of Inertia, and Damping Factors for Circular Disk $\tilde{b}=1$.

	M_y	N_y	I_x	H_x	I_z	H_z
$a = \pi/6$	2.242	0.990	0.351	0.032	0.335	0.030
	(2.194)	(0.977)	(0.262)	(0.029)	(0.194)	(0.0[illegible]3)
$a = \pi/5$	2.129	1.016	0.415	0.064	0.395	0.060
	(2.082)	(1.006)	(0.306)	(0.055)	(0.232)	(0.044)
$a = \pi/4$	1.987	1.026	0.536	0.161	0.509	0.150
	(1.943)	(1.021)	(0.385)	(0.130)	(0.299)	(0.101)
$a = \pi/3$	1.809	1.002	0.677	0.651	0.650	0.599
	(1.774)	(1.007)	(0.483)	(0.458)	(0.387)	(0.338)
$a = 2\pi/5$	1.706	0.968	0.126	1.136	0.163	1.078
	(1.659)	(0.970)	(0.069)	(0.874)	(0.125)	(0.683)
$a = \pi/2$	1.593	0.910	-0.497	0.602	-0.469	0.601
	(1.582)	(0.923)	(-0.366)	(0.575)	(-0.277)	(0.543)
$a = 2\pi/3$	1.473	0.811	-0.381	0.167	-0.370	0.168
	(1.489)	(0.809)	(-0.336)	(0.147)	(-0.316)	(0.178)
$a = \pi$	1.341	0.628	-0.127	0.031	-0.138	0.009
	(1.342)	(0.533)	(-0.151)	(0.001)	(-0.190)	(0.017)

Table 3

Added Mass, Added Moment of Inertia, and Damping Factors for Elliptic Disk $\tilde{b}=1/4$.

	M_y	N_y	I_x	H_x	I_z	H_z
$a = \pi/6$	0.298	0.099	0.0012	0.00001	0.052	0.002
	(0.294)	(0.097)	(0.0018)	(0.00001)	(0.040)	(0.001)
$a = \pi/5$	0.288	0.108	0.0013	0.00001	0.057	0.003
	(0.284)	(0.106)	(0.0018)	(0.00001)	(0.044)	(0.003)
$a = \pi/4$	0.274	0.119	0.0014	0.00002	0.065	0.007
	(0.270)	(0.116)	(0.0018)	(0.00002)	(0.051)	(0.006)
$a = \pi/3$	0.251	0.129	0.0015	0.00005	0.084	0.020
	(0.248)	(0.127)	(0.0018)	(0.00005)	(0.065)	(0.016)

Table 3

Added Mass, Added Moment of Inertia, and Damping Factors for Elliptic Disk $\tilde{b}=1/4$.

	M_y	N_y	I_x	H_x	I_z	H_z
$a=2\pi/5$	0.235	0.133	0.0017	0.00010	0.100	0.043
	(0.233)	(0.131)	(0.0018)	(0.00008)	(0.078)	(0.033)
$a=\pi/2$	0.215	0.135	0.0019	0.00020	0.102	0.112
	(0.213)	(0.133)	(0.0019)	(0.00017)	(0.082)	(0.083)
$a=2\pi/3$	0.187	0.132	0.0025	0.00058	-0.041	0.139
	(0.187)	(0.131)	(0.0021)	(0.00044)	(-0.020)	(0.121)
$a=\pi$	0.154	0.117	0.0032	0.00312	-0.037	0.014
	(0.155)	(0.118)	(0.0023)	(0.00184)	(-0.040)	(0.018)

Table 4

Added Mass, Added Moment of Inertia, and Damping Factors for Elliptic Disk $\tilde{b}=1/8$.

	M_y	N_y	I_x	H_x	I_z	H_z
$a=\pi/6$	0.105	0.029	0.00011	0.0000001	0.020	0.0004
	(0.107)	(0.028)	(0.00027)	(0.0000001)	(0.018)	(0.0003)
$a=\pi/5$	0.102	0.032	0.00011	0.0000002	0.021	0.001
	(0.104)	(0.031)	(0.00027)	(0.0000002)	(0.019)	(0.001)
$a=\pi/4$	0.098	0.036	0.00011	0.0000003	0.023	0.002
	(0.100)	(0.035)	(0.00026)	(0.0000003)	(0.022)	(0.001)
$a=\pi/3$	0.090	0.041	0.00012	0.0000008	0.027	0.004
	(0.092)	(0.039)	(0.00025)	(0.0000007)	(0.026)	(0.003)
$a=2\pi/5$	0.085	0.043	0.00013	0.0000013	0.031	0.008
	(0.087)	(0.041)	(0.00025)	(0.0000013)	(0.030)	(0.007)
$a=\pi/2$	0.077	0.045	0.00013	0.0000027	0.037	0.018
	(0.079)	(0.043)	(0.00024)	(0.0000025)	(0.035)	(0.016)
$a=2\pi/3$	0.066	0.044	0.00015	0.0000066	0.027	0.046
	(0.068)	(0.042)	(0.00023)	(0.0000059)	(0.027)	(0.043)
$a=\pi$	0.051	0.039	0.00020	0.0000252	-0.017	0.023
	(0.053)	(0.037)	(0.00021)	(0.0000208)	(-0.016)	(0.022)

fine lattice yield much improved results since I_x and I_z, and H_x and H_z must be equal in the case of a circular disk.

In Figure 1 and Figure 2 the dependence of $\frac{\pi\bar{a}^2}{A}M_y$ and $\frac{\pi\bar{a}^2}{A}N_y$ on the parameter a are presented. In Figure 3 and Figure 4 the quantities $\frac{\pi\bar{a}^2}{A}I_x$ and $\frac{\pi\bar{a}^2}{A}H_x$ are plotted as functions of the parameter a. Similary, the quantities $\frac{\pi\bar{a}^2}{A}I_z$ and $\frac{\pi\bar{a}^2}{A}H_z$ are plotted in Figure 5 and Figure 6. The multiplication factor $\pi\bar{a}^2/A$ for the ordinates represents the ratio of the area of a circle having the half length of the ship $\bar{a}$ as the radius to the area of the water plane of the ship under consideration, and was introduced in order to make the curves comparable. Note that these curves are obtained from the results of the fine lattice.

The curves for the circular disk $\tilde{b}=1$ in Figure 1 and Figure 2 compare very closely to the corresponding curves in Figure 6 and Figure 7 in [5] which were obtained by treating the circular disk as an axial symmetric two-dimensional configuration.

To ascertain the accuracy of the results represented by the curves for $\tilde{b}=1/4$ and $\tilde{b}=1/8$ in Figure 2, the normalized damping factor N'_y were computed by the following formula based on the strip method,

(VI.1) $$N'_y = 2\int_0^1 \left(\frac{\bar{a}}{\bar{b}}\right)^2 (1-\tilde{x}^2)\, N_2\left(\frac{\bar{a}}{\bar{b}}\, a\sqrt{1-\tilde{x}^2}\right) d\tilde{x} \,,$$

where the value $N_2(\frac{\bar{a}}{\bar{b}}\, a\sqrt{1-\tilde{x}^2})$ are taken from Figure 4 in [5] using the relation, $N_2(\frac{\bar{a}}{\bar{b}}\, a\sqrt{1-\tilde{x}^2}) = \frac{1}{a^2}\left|\zeta/\zeta_0\right|^2$. As shown in Table 5, for low frequency the strip method does not yield a satisfactory results. For this range the present method should give accurate results.

Table 5

Comparison of Damping Factor of Elliptic Disks for Heave Evaluated by Integral Equation Method and Strip Method.

	For $\tilde{b}=1/4$			For $\tilde{b}=1/8$		
	N_y^o	N_y^f	N_y'	N_y^o	N_y^f	N_y'
$a = \pi/3$	0.127	0.129	0.183	(Not Available)		
$a = \pi/2$	0.133	0.135	0.166	0.043	0.044	0.049
$a = 2\pi/3$	0.131	0.132	0.153	0.042	0.044	0.046

Here in Table 5, N_y^o denotes the value obtained by the use of the original lattice and N_y^f denotes that by the use of the fine lattice.

An additional check for the results of the heave of a circular disk can be made by comparing the values of $f_r(\tilde{x},\tilde{z})$ and $f_i(\tilde{x},\tilde{z})$ at pivotal points of the equal radial distances. We compare these values at the tip and the half radial distance on the $\tilde{x}$-axis with those at the tip and at the half radial distance on the $\tilde{z}$-axis in Table 6 using the results of the fine lattice. At higher frequency the agreement was found to be unsatisfactory presumably due to the use of non-square grid which is primarily designed for the elliptic disk.

Table 6

Comparison of Real and Imaginary Parts of Density for Heave at Pivotal Points of Equal Radial Distances on Circular Disk.

	$f_r(\tilde{x},\tilde{z})$				$f_i(\tilde{x},\tilde{z})$			
At	Tip on $\tilde{x}$	Tip on $\tilde{z}$	1/2 on $\tilde{x}$	1/2 on $\tilde{z}$	Tip on $\tilde{x}$	Tip on $\tilde{z}$	1/2 on $\tilde{x}$	1/2 on $\tilde{z}$
$a = \pi/6$	0.727	0.712	0.587	0.596	-0.185	-0.181	-0.160	-0.163
$a = \pi/5$	0.694	0.678	0.526	0.536	-0.232	-0.226	-0.196	-0.200
$a = \pi/4$	0.655	0.635	0.442	0.452	-0.301	-0.292	-0.245	-0.251
$a = \pi/3$	0.604	0.579	0.312	0.320	-0.409	-0.393	-0.315	-0.324
$a = 2\pi/5$	0.572	0.543	0.212	0.218	-0.490	-0.468	-0.361	-0.373
$a = \pi/2$	0.535	0.498	0.063	0.064	-0.604	-0.570	-0.417	-0.433
$a = 2\pi/3$	0.488	0.435	-0.185	-0.198	-0.774	-0.719	-0.482	-0.505
$a = \pi$	0.424	0.331	-0.678	-0.729	-1.048	-0.954	-0.523	-0.558

We remark that as the frequency of the forced oscillation tends to zero, that is $a \to 0$, M_y becomes a constant and N_y being $a\tilde{S}^2$ the damping factor will vanish. In two dimensional case however, M_y being $O(\log a)$ the added mass tends to infinity while N_y becomes a constant. In Figure 2, as a tends to zero the ordinate becomes,

$$\lim_{a \to 0} \frac{\pi\bar{a}^2}{A} N_y = \frac{\pi\bar{a}^2}{A}(a\tilde{S}^2) = \frac{\bar{b}}{\bar{a}} \pi^2 a$$

where $A = \pi\bar{a}\bar{b}$, and $S = \dfrac{\pi\bar{a}\bar{b}}{\bar{a}^2}$ for the elliptic disk. Hence at the origin the slopes are,

$$\pi^2 = 9.8696 \qquad \text{for } \tilde{b}=1,$$
$$1/4\ \pi^2 = 2.4674 \qquad \text{for } \tilde{b}=1/4,$$
$$1/8\ \pi^2 = 1.2337 \qquad \text{for } \tilde{b}=1/8, \quad \text{respectively.}$$

These slopes are shown in Figure 2 by the straight lines. It is to be observed that the computed results deviate very rapidly from the low frequency approximation.

For the roll and pitch, the results comparable to those presented in Figure 3 to Figure 6 cannot be found in the existing literatures. For the case of circular disk $\tilde{b}=1$, realizing that roll and pitch are equivalent if the disk is turned around by the right angle, we compare the values of $f_r(\tilde{x},\tilde{z})$ and $f_i(\tilde{x},\tilde{z})$ for roll and pitch at pivotal points of the equal radial distance. More precisely these values for pitch at the tip and at the half radial distance on $\tilde{x}$-axis are compared with those for roll at the tip and at the half radial distance on $\tilde{z}$-axis in Table 7 using the results of the fine lattice. A bad agreement was found at $a = \pi$ again probably due to the geometrically unsuitable lattice in use.

Table 7

Comparison of Real and Imaginary Parts of Density for Roll and Pitch at Pivotal Points of Equal Radial Distances on Circular Disk.

	$f_r(\tilde{x},\tilde{z})$				$f_i(\tilde{x},\tilde{z})$			
At	Tip on $\tilde{x}$, (Roll)	Tip on $\tilde{z}$, (Pitch)	1/2 on $\tilde{x}$, (Roll)	1/2 on $\tilde{z}$, (Pitch)	Tip on $\tilde{x}$, (Roll)	Tip on $\tilde{z}$, (Pitch)	1/2 on $\tilde{x}$, (Roll)	1/2 on $\tilde{z}$ (Pitch)
$a = \pi/6$	-1.144	-1.153	-0.669	-0.657	-0.020	-0.020	-0.012	-0.011
$a = \pi/5$	-1.208	-1.218	-0.739	-0.722	-0.046	-0.044	-0.029	-0.027
$a = \pi/4$	-1.339	-1.350	-0.886	-0.859	-0.138	-0.134	-0.095	-0.089
$a = \pi/3$	-1.522	-1.545	-1.193	-1.150	-0.671	-0.649	-0.560	-0.514
$a = 2\pi/5$	-0.905	-0.977	-0.795	-0.812	-1.259	-1.260	-1.282	-1.206
$a = \pi/2$	-0.126	-0.146	0.089	0.060	-0.642	-0.683	-1.006	-0.986
$a = 2\pi/3$	-0.265	-0.268	0.259	0.239	-0.039	-0.049	-0.568	-0.553
$a = 5\pi/6$	-0.584	-0.558	0.479	0.444	0.172	0.209	-0.396	-0.378
$a = \pi$	-0.748	-1.026	0.561	0.604	-0.371	-0.215	0.479	0.276

We further observe that as the frequency of the forced oscillation tends to zero, that is $a \to 0$, I_x and I_z become constants, while H_x and H_z being $O(a^2)$ the damping factors will vanish.

The computation work is performed with the IBM 7090 data processing system at the Westinghouse Electric Corporation in East Pittsburgh Works. The program for the computation is coded into the Fortran language.

Figure 1

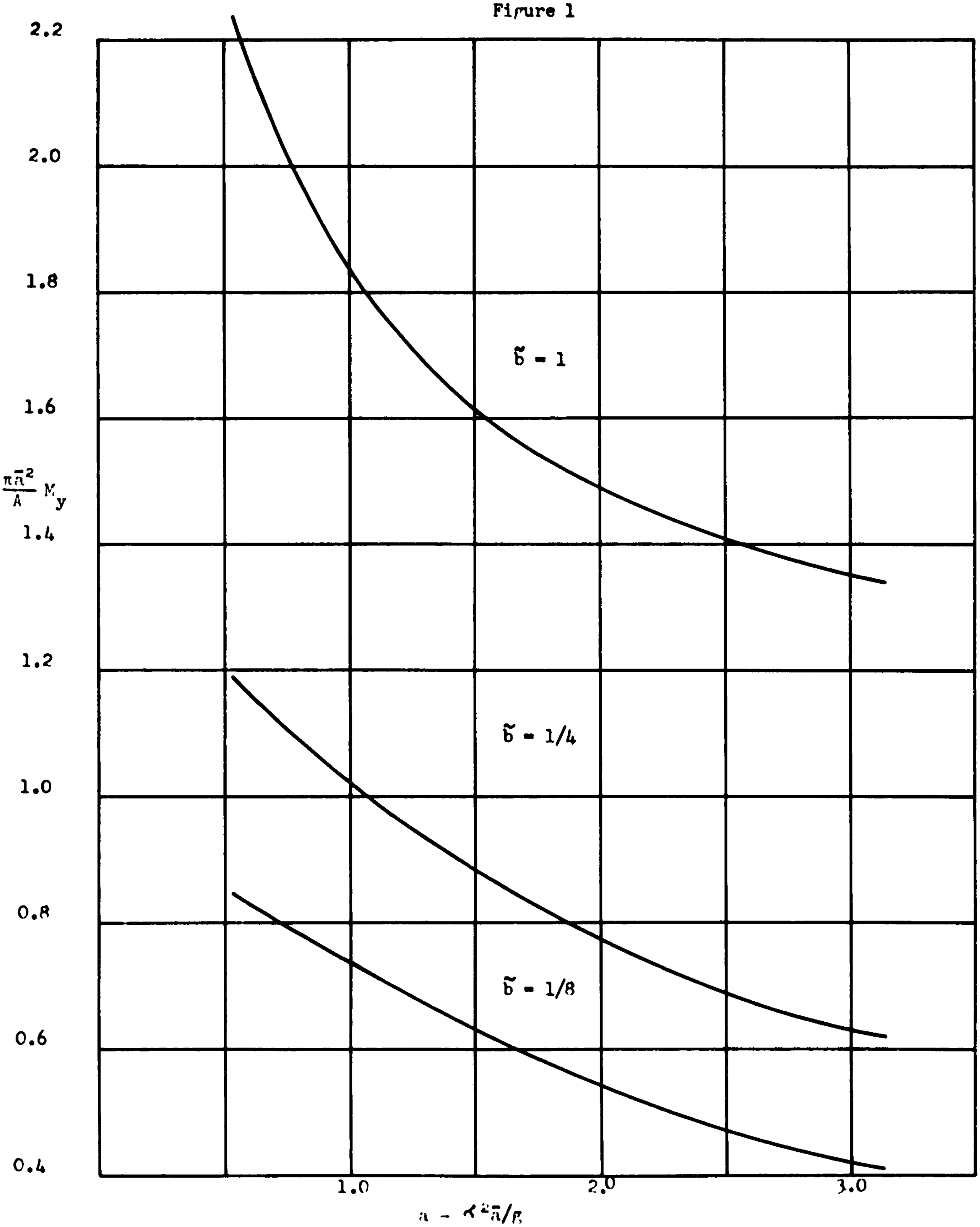

2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
$\frac{\pi \bar{a}^2}{A} M_y$
$\tilde{b} = 1$
$\tilde{b} = 1/4$
$\tilde{b} = 1/8$
1.0
2.0
3.0
$a = \alpha^2 \bar{a}/\beta$

Figure 2

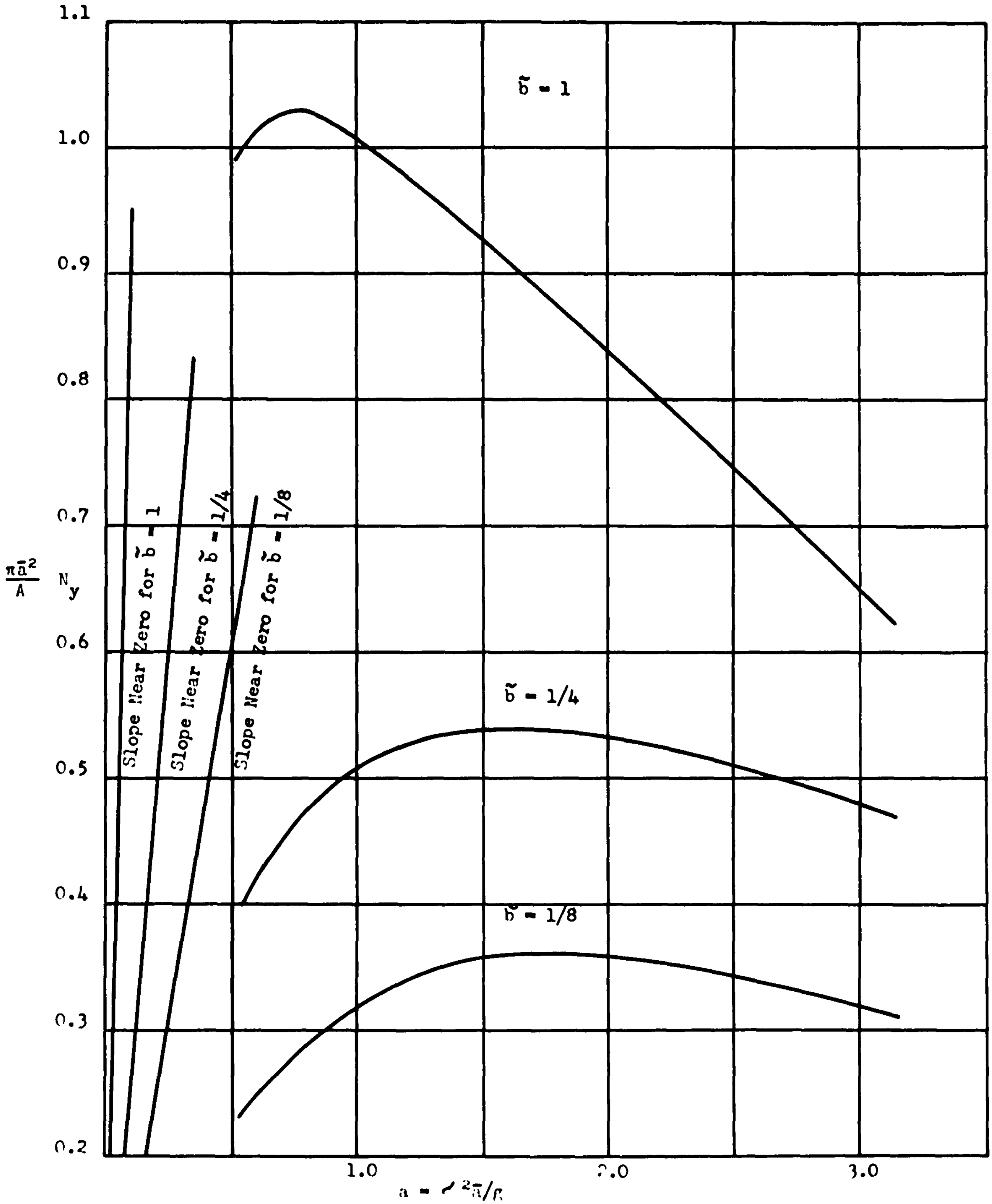

1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
$\frac{\pi \bar{a}^2}{A}$ N_y
$\tilde{b} = 1$
$\tilde{b} = 1/4$
$\tilde{b} = 1/8$
Slope Near Zero for $\tilde{b} = 1$
Slope Near Zero for $\tilde{b} = 1/4$
Slope Near Zero for $\tilde{b} = 1/8$
1.0
2.0
3.0

Figure 3

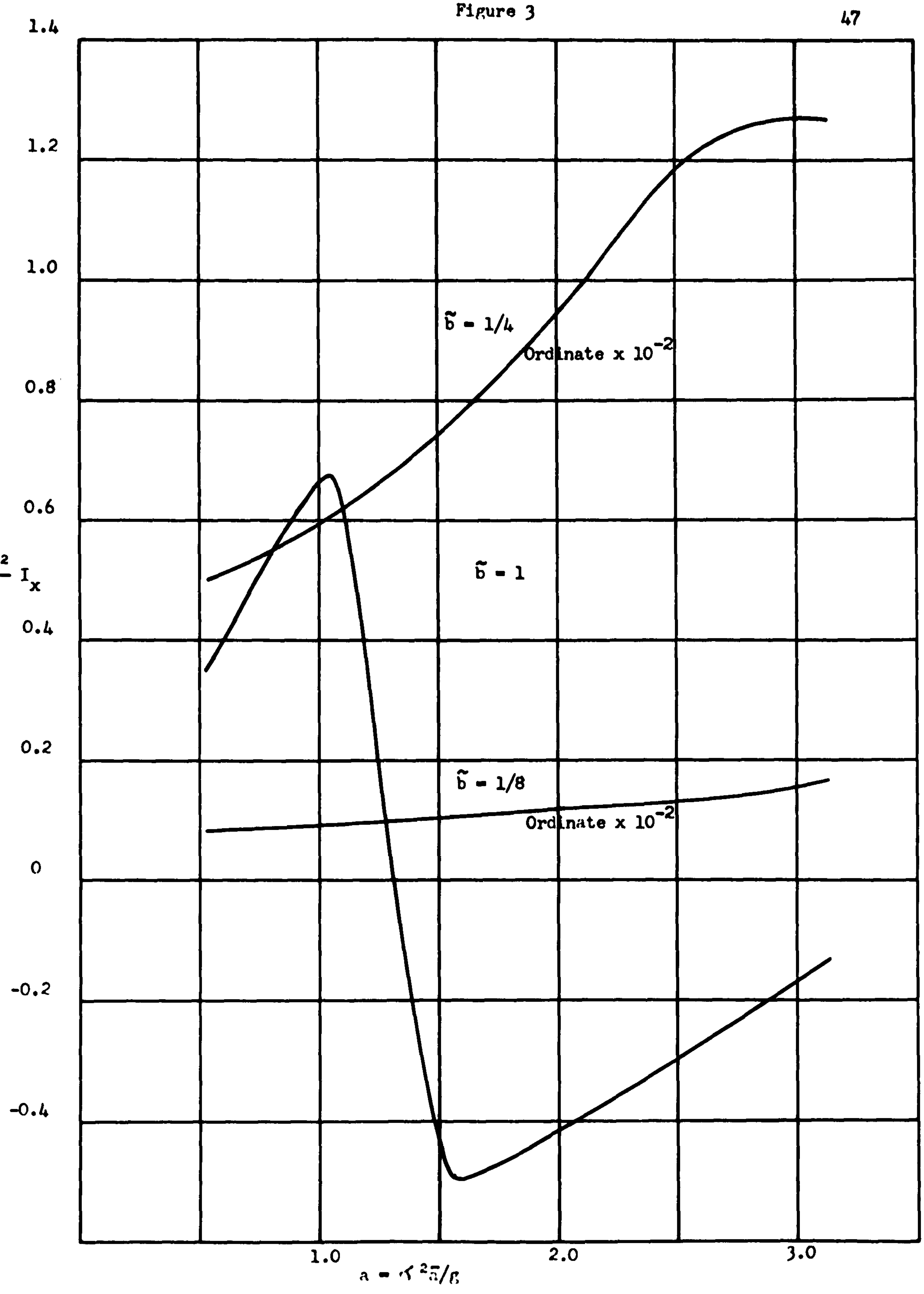

1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
$\frac{\pi\tilde{a}^2}{A} I_x$
$\tilde{b} = 1/4$
Ordinate x 10^{-2}
$\tilde{b} = 1$
$\tilde{b} = 1/8$
Ordinate x 10^{-2}
1.0
2.0
3.0

Figure 4

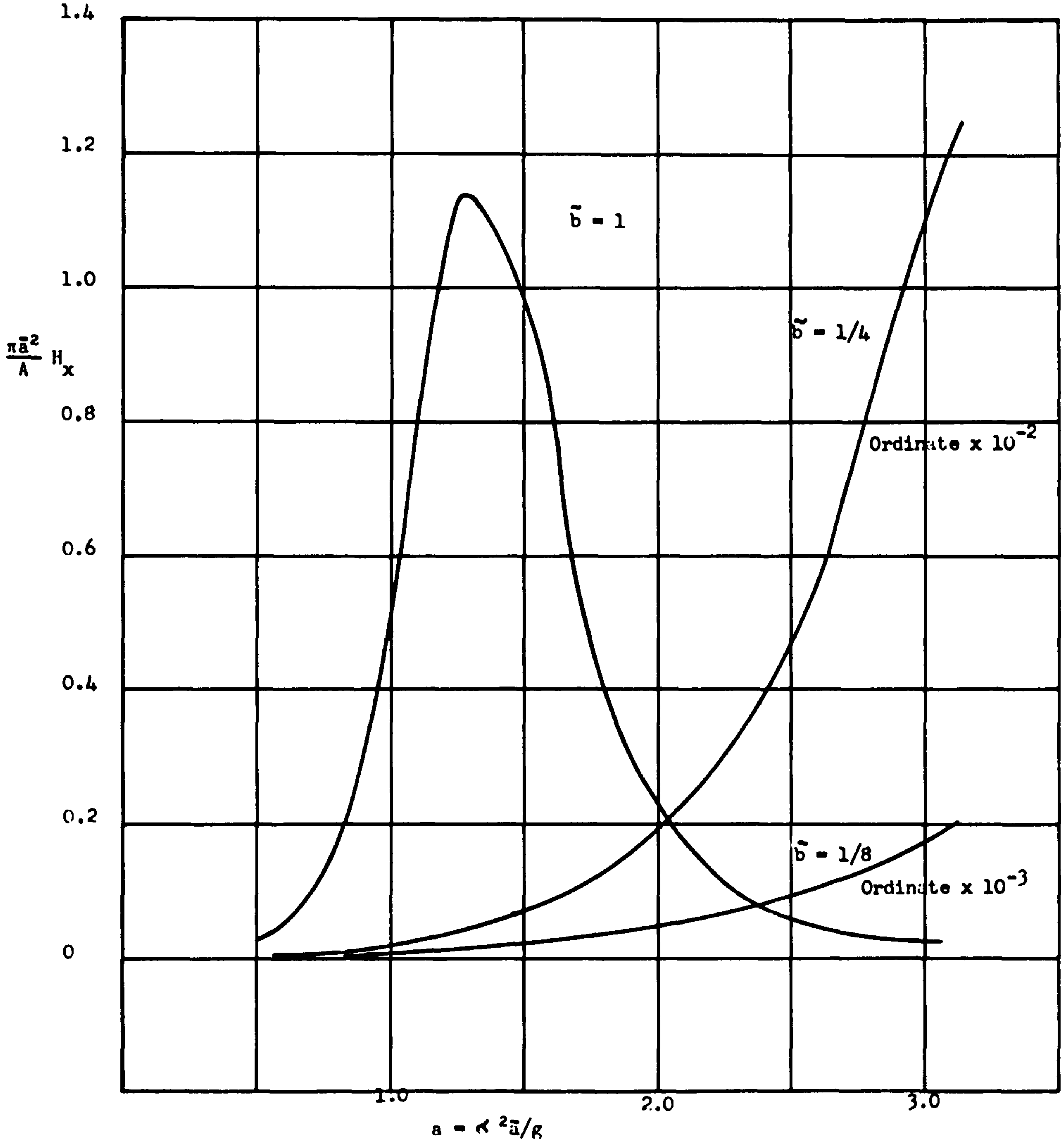

1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
$\frac{\pi \bar{a}^2}{A} H_x$
$\tilde{b} = 1$
$\tilde{b} = 1/4$
Ordinate x 10^{-2}
$\tilde{b} = 1/8$
Ordinate x 10^{-3}
1.0
2.0
3.0
$a = \alpha^2 \bar{a}/g$

Figure 5

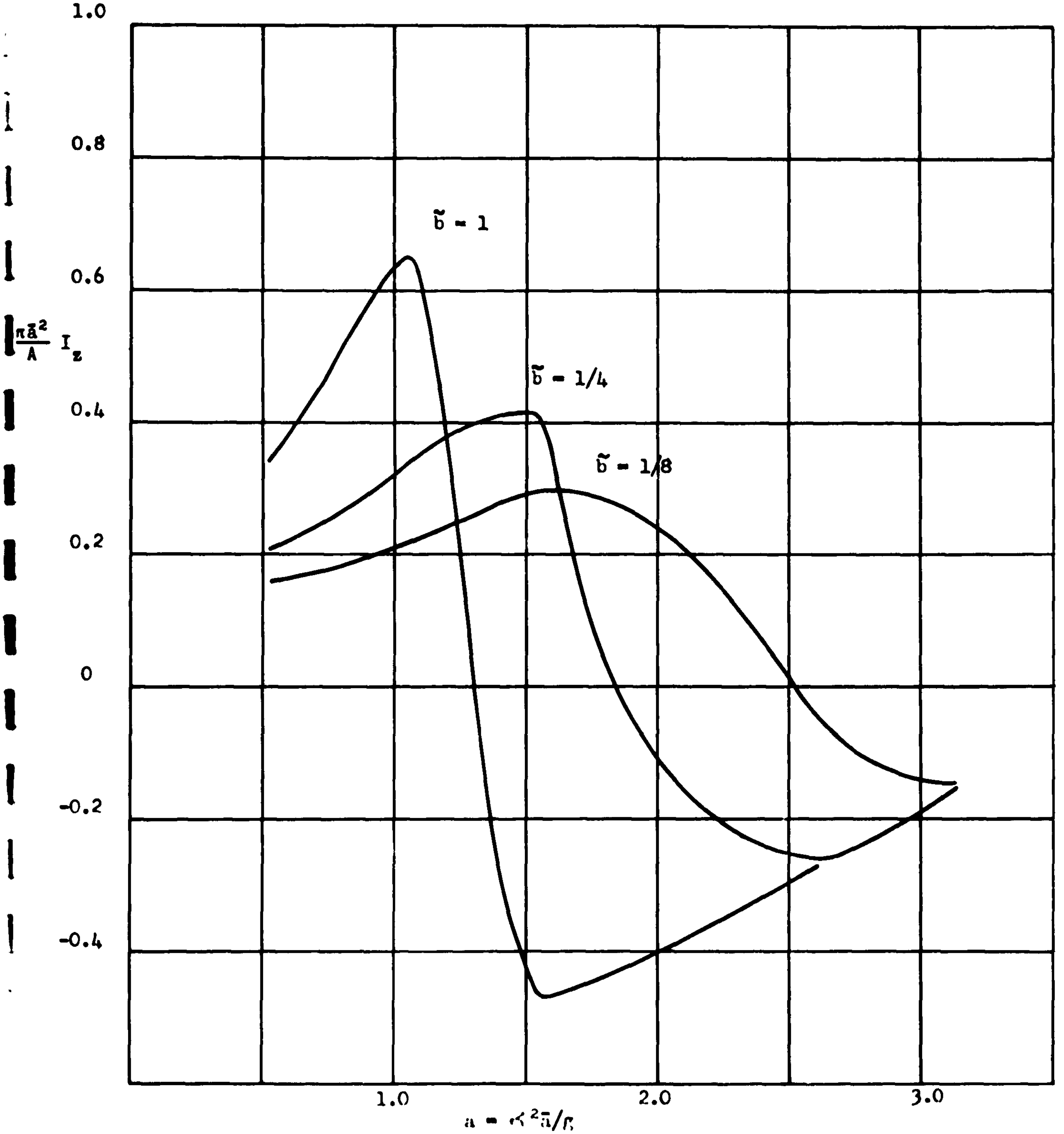
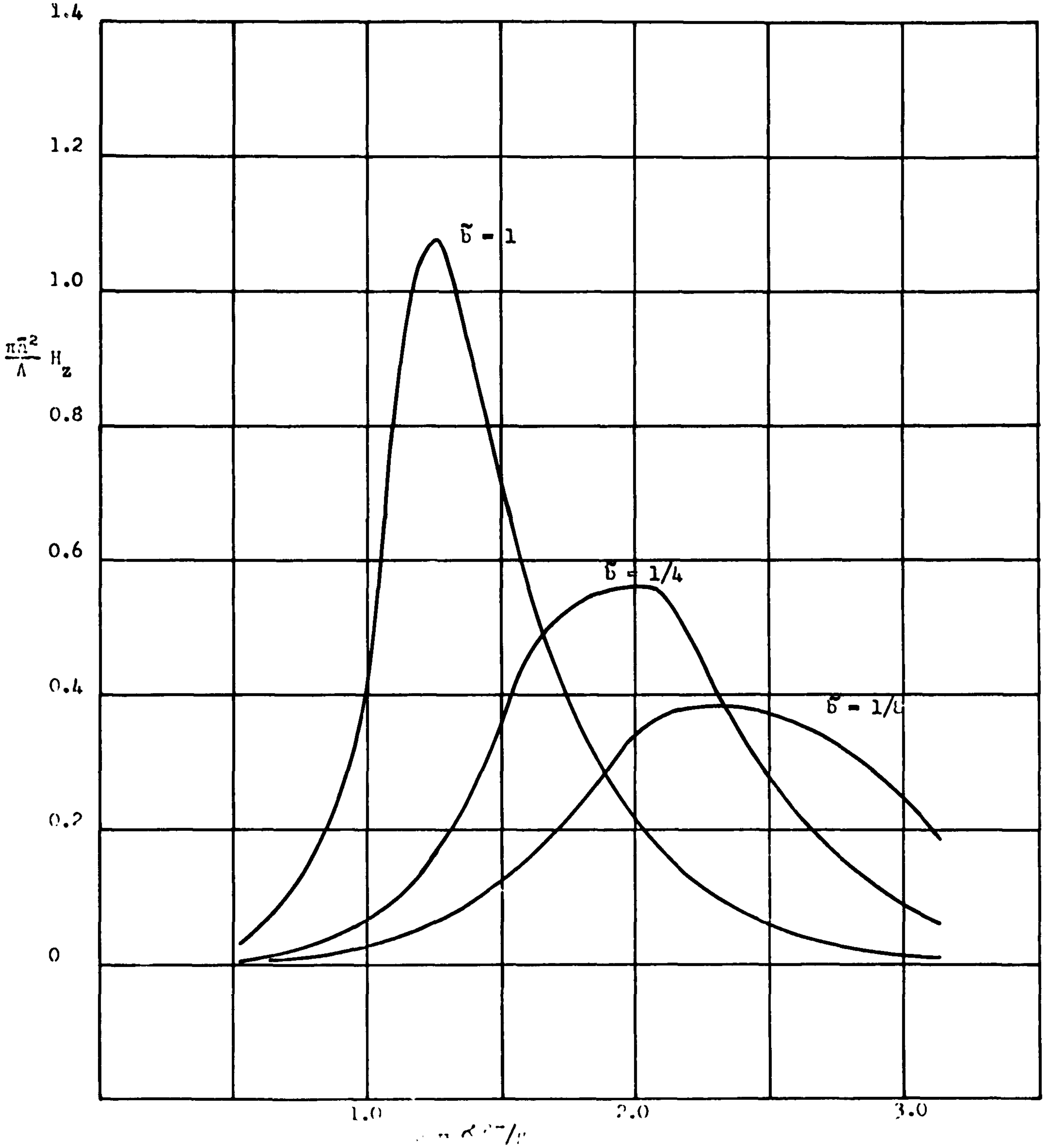

1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
$\frac{\pi \bar{a}^2}{A} I_z$
$\tilde{b} = 1$
$\tilde{b} = 1/4$
$\tilde{b} = 1/8$
1.0
2.0
3.0

Figure 6


1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
$\frac{\pi \bar{a}^2}{\Lambda} H_z$
$\tilde{b} = 1$
$\tilde{b} = 1/4$
1.0
2.0
3.0

BIBLIOGRAPHY

1. John, F., On the motion of floating bodies, II. Commun. on Pure and Applied Math., Vol. III, No. 1, March 1950.

2. Stoker and Peters, The motion of a ship as a floating rigid body in a seaway, Commun. on Pure and Applied Math., Vol. X, No. 3, 1957.

3. Haskind, M. D., The hydrodynamical theory of the oscillation of a ship in waves, Russian translation, 1946.

4. MacCamy, R. C., On the scattering of water waves by a circular disk, Archive for Rational Mechanics and Analysis, Vol. 8, No. 2, 1961.

5. MacCamy, R. C., On the heaving motion of cylinders of shallow draft, Journal of Ship Research (To appear).

6. MacCamy, R. C., Asymptotic development for a boundary value problem containing a parameter, Quart. of Applied Math., Vol. XVII, No. 2, July 1959.

7. Grobner and Hofreiter, Intergraltafeln, Springer-Verlog, 1949.

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